

## Annual Review of Chaos Theory, Bifurcations and Dynamical Systems



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# Annual Review of Chaos Theory, Bifurcations 

# Annual Review of Chaos Theory, Bifurcations and Dynamical Systems 

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Editorial office (Algeria office): Annual Review of Chaos Theory, Bifurcations and Dynamical Systems (ARCTBDS) Dr. Zeraoulia Elhadj, Department of Mathematics, University of Tébessa, (12002), Algeria, e-mail: zeraoulia@mail.univ-tebessa.dz and zelhadj12@yahoo.fr.

# Annual Review of Chaos Theory, Bifurcations 

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Table of Contents
Solving Nonlinear Equations Using Two-Step Optimal MethodsM.A. Hafiz, M. S. M. Bahgat01-11
A Dynamical Analysis of an Autocatalytic Model Lakshmi Burra, Uma Maheswari ..... 12-22
Chaos in the Planar Two-Body Coulomb Problem with a Uniform Magnetic Field Vladimir Zhdankins, J. C. Sprott ..... 23-33
On a Conjecture of Trichotomy and Bifurcation in a Third Order Rational Difference Equation Xianyi Li, Cheng Wang

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# Solving Nonlinear Equations Using Two-Step Optimal Methods 

M.A. Hafiz<br>Department of mathematics, Faculty of Science and arts, Najran University, Saudi Arabia.<br>e-mail: admin@mhafiz.net<br>M. S. M. Bahgat<br>Department of mathematics, Faculty of Science, Minia University, Minia 61111 Egypt. e-mail: msmbahgat66@hotmail.com

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## 1 Introduction

Finding iterative methods for solving nonlinear equations is an important area of research in numerical analysis at it has interesting applications in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x)=$ 0 . Due to their importance, several numerical methods have been suggested and analyzed under certain condition. These numerical methods have been constructed using different techniques such as Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variational iteration method, and decomposition method ( see [1-9]). In this study we describe new iterative free from second derivative to find a simple root rof a nonlinear equation. In the implementation of the method of Noor et al. [10], one has to evaluate the second derivative of the function, which is a serious drawback of these methods. To overcome these drawbacks, we modify the predictor-corrector Halley method by replacing the second derivatives of the function by its suitable finite difference scheme. As we will show, the obtained two-step methods are of fourth-order of convergence and require three evaluations of the function $f(x)$. The procedure of removing the derivatives usually increases the number of functional evaluations per iteration. Commonly in the literature the efficiency of an iterative method is measured by the efficiency index defined
as $I \approx p^{1 / d}(\operatorname{see}[11])$, where $p$ is the order of convergence and $d$ is the total number of functional evaluations per step. Therefore these methods have efficiency index $4^{1 / 3} \approx$ 1.5874 that is, the new family of methods reaches the optimal order of convergence four, which is higher than $2^{1 / 2} \approx 1.4142$ of the Steffensen's method (SM) (see [12]), $3^{1 / 4} \approx$ 1.3161 of the DHM method (see [13]), $9^{1 / 5} \approx 1.552$ of the method [14] and our methods are equivalent to the LZM [15] and CTM [16]. We prove that our methods are of fourthorder convergence and present the comparison of these new methods with other methods. Several examples are given to illustrate the efficiency and performance of these methods.

## 2 Iterative methods

For the sake completeness, we recall Newton, Halley, Traub, and Homeier methods. These methods as follows:
Algorithm 2.1. For a given $x_{0}$, find the approximate solution $x_{n+1}$ by the iterative scheme

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}
$$

It is well known that algorithm 2.1 has a quadratic convergence.
Algorithm 2.2. For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative scheme

$$
x_{n+1}=x_{n}-\frac{2 f\left(x_{n}\right) f^{\prime}\left(x_{n}\right)}{2 f^{\prime 2}\left(x_{n}\right)-f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)} .
$$

This is known as Halley's method and has cubic convergence [6].
Algorithm 2.3. For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{aligned}
& y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}, \\
& x_{n+1}=y_{n}-\frac{f\left(y_{n}\right)}{f^{\prime}\left(y_{n}\right)} .
\end{aligned}
$$

Algorithm 2.3 is called the predictor-corrector Newton method (PCN) and has fourthorder convergence (see [16]). Homeier [17] derived the following cubically convergent iteration scheme
Algorithm 2.4. For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{gathered}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}, \\
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{2}\left(\frac{1}{f^{\prime}\left(x_{n}\right)}+\frac{1}{f^{\prime}\left(y_{n}\right)}\right) .
\end{gathered}
$$

The first and the second derivatives with respect to $y$, which may create some problems. To overcome this drawback, several authors have developed involving only the first derivatives. This idea plays a significant part in developing our new iterative methods free from first and second derivatives with respect to $y$. To be more precise, we now approximate $f^{\prime}\left(y_{n}\right)$, to reduce the number of evaluations per iteration by a combination of already
known data in the past steps. Toward this end, an estimation of the function $P_{1}(t)$ is taken into consideration as follows

$$
\begin{aligned}
& P_{1}(t)=a+b\left(t-x_{n}\right)+c\left(t-x_{n}\right)^{2} \\
& P_{1}^{\prime}(t)=b+2 c\left(t-x_{n}\right)
\end{aligned}
$$

By substituting in the known values

$$
\begin{aligned}
& P_{1}\left(y_{n}\right)=f\left(y_{n}\right)=a+b\left(y_{n}-x_{n}\right)+c\left(y_{n}-x_{n}\right)^{2} \\
& P_{1}^{\prime}\left(y_{n}\right)=f^{\prime}\left(y_{n}\right)=b+2 c\left(y_{n}-x_{n}\right) \\
& P_{1}\left(x_{n}\right)=f\left(x_{n}\right)=a \\
& P_{1}^{\prime}\left(x_{n}\right)=f^{\prime}\left(x_{n}\right)=b
\end{aligned}
$$

we could easily obtain the unknown parameters. Thus we have

$$
\begin{equation*}
f^{\prime}\left(y_{n}\right)=2\left(\frac{f\left(y_{n}\right)-f\left(x_{n}\right)}{y_{n}-x_{n}}\right)-f^{\prime}\left(x_{n}\right)=P_{1}\left(x_{n}, y_{n}\right) \tag{1}
\end{equation*}
$$

At this time, it is necessary to approximate $f^{\prime \prime}\left(y_{n}\right)$, with a combination of known values Accordingly, we take account of an interpolating polynomial

$$
P_{2}(t)=a+b\left(t-x_{n}\right)+c\left(t-x_{n}\right)^{2}+d\left(t-x_{n}\right)^{3}
$$

and also consider that this approximation polynomial satisfies the interpolation conditions $f\left(x_{n}\right)=P_{2}\left(x_{n}\right), f\left(y_{n}\right)=P_{2}\left(y_{n}\right), f^{\prime}\left(x_{n}\right)=P_{2}^{\prime}\left(x_{n}\right)$ and $f^{\prime}\left(y_{n}\right)=P_{2}^{\prime}\left(y_{n}\right)$, By substituting the known values in $P_{2}(t)$ we have a system of three linear equations with three unknowns. By solving this system and simplifying we have

$$
\begin{equation*}
f^{\prime \prime}\left(y_{n}\right)=\frac{2}{y_{n}-x_{n}}\left(\frac{f\left(y_{n}\right)-f\left(x_{n}\right)}{y_{n}-x_{n}}-f^{\prime}\left(x_{n}\right)\right)=P_{2}\left(x_{n}, y_{n}\right) \tag{2}
\end{equation*}
$$

then algorithm 2.3 can be written in the form of the following algorithm.
Algorithm 2.5. For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{aligned}
& y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}, \\
& x_{n+1}=y_{n}-\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)} .
\end{aligned}
$$

This method has fourth-order convergence is called Khattri method (KM) [18]. Now using equations (1) and (2) to suggest the following new iterative methods for solving nonlinear equation, and use Algorithm 2.1 as predictor and Algorithm 2.2 as a corrector. It is established that the following new methods have convergence order four, which will denote by Hafiz and Bahgat Methods (HBM1-HBM5).
HBM1: For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{gathered}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}, \\
x_{n+1}=y_{n}-\frac{2 f\left(y_{n}\right) P_{1}\left(x_{n}, y_{n}\right)}{2 P_{1}^{2}\left(x_{n}, y_{n}\right)-f\left(y_{n}\right) P_{2}\left(x_{n}, y_{n}\right)} .
\end{gathered}
$$

(HBM1) is called the new two-step modified Halley's method free from second and first derivative with respect to $y$, for solving nonlinear equation $f(x)=0$.

HBM2: For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{gathered}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)} \\
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{2}\left[\frac{1}{f^{\prime}\left(x_{n}\right)}+\frac{1}{P_{1}\left(x_{n}, y_{n}\right)}\right] .
\end{gathered}
$$

HBM3: For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{gathered}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}, \\
x_{n+1}=y_{n}-\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}-\frac{f^{2}(y) P_{2}\left(x_{n}, y_{n}\right)}{2 P_{1}^{3}\left(x_{n}, y_{n}\right)} .
\end{gathered}
$$

If $P_{2}\left(x_{n}, y_{n}\right)=0$, then HBM1 and HBM3 deduces Algorithm 2.5.
HBM4: For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{gathered}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}, \\
x_{n+1}=y_{n}-\left[1+\left(\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\right)^{2}\right] \frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)} .
\end{gathered}
$$

HBM5: For a given $x_{0}$, compute approximates solution $x_{n+1}$ by the iterative schemes

$$
\begin{gathered}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)} \\
x_{n+1}=y_{n}+f\left(y_{n}\right)\left[\frac{1}{f^{\prime}\left(x_{n}\right)}-\frac{4}{f^{\prime}\left(x_{n}\right)+P_{1}\left(x_{n}, y_{n}\right)}\right]
\end{gathered}
$$

Let us remark that, in terms of computational cost, the developed methods (HBM1HBM5) require only three functional evaluations per step. So, they have efficiency indices $4^{1 / 3} \approx 1.5874$, that is, the new family of methods (HBM1- HBM5) reaches the optimal order of convergence four, conjectured by Kung and Traub [16].

## 3 Convergence analysis

Let us now discuss the convergence analysis of the above mentioned methods (HBM1HBM5).

Theorem 3.1: Let $r$ be a sample zero of sufficient differentiable function $f: \subseteq R \rightarrow R$ for an open interval $I$. If $x_{0}$ is sufficiently close to $r$, then the two-step method defined by (HBM1)has fourth-order convergence.

Proof. Consider to

$$
\begin{equation*}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}, \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
x_{n+1}=y_{n}-\frac{2 f\left(y_{n}\right) P_{1}\left(x_{n}, y_{n}\right)}{2 P_{1}^{2}\left(x_{n}, y_{n}\right)-f\left(y_{n}\right) P_{2}\left(x_{n}, y_{n}\right)} . \tag{4}
\end{equation*}
$$

Let $r$ be a simple zero of $f$. Since $f$ is sufficiently differentiable, by expanding $f\left(x_{n}\right)$ and $f^{\prime}\left(x_{n}\right)$ about $r$, we get
$f\left(x_{n}\right)=f(r)+\left(x_{n}-r\right) f^{\prime}(r)+\frac{\left(x_{n}-r\right)^{2}}{2!} f^{(2)}(r)+\frac{\left(x_{n}-r\right)^{3}}{3!} f^{(3)}(r)+\frac{\left(x_{n}-r\right)^{4}}{4!} f^{(4)}(r)+\cdots$,
then

$$
\begin{equation*}
f\left(x_{n}\right)=f^{\prime}(r)\left[e_{n}+c_{2} e_{n}^{2}+c_{3} e_{n}^{3}+c_{4} e_{n}^{4}+\cdots\right] \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
f^{\prime}\left(x_{n}\right)=f^{\prime}(r)\left[1+2 c_{2} e_{n}+3 c_{3} e_{n}^{2}+4 c_{4} e_{n}^{3}+5 c_{5} e_{n}^{4}+\cdots\right], \tag{6}
\end{equation*}
$$

where $c_{k}=\frac{1}{k!} \frac{f^{(k)}(r)}{f^{\prime}(r)}, k=2,3, \ldots$ and $e_{n}=x_{n}-r$.
Now from (5) and (6), we have

$$
\begin{equation*}
\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=e_{n}-c_{2} e_{n}^{2}+2\left(c_{2}^{2}-c_{3}\right) e_{n}^{3}+\left(7 c_{2} c_{3}-4 c_{2}^{3}+3 c_{4}\right) e_{n}^{4}+\cdots \tag{7}
\end{equation*}
$$

From (3) and (7), we get

$$
\begin{equation*}
y_{n}=r+c_{2} e_{n}^{2}+2\left(c_{3}-c_{2}^{2}\right) e_{n}^{3}+\left(-7 c_{2} c_{3}+4 c_{2}^{3}+3 c_{4}\right) e_{n}^{4}+\cdots, \tag{8}
\end{equation*}
$$

From (8), we get,

$$
\begin{align*}
f\left(y_{n}\right)= & f^{\prime}(r)\left[\left(y_{n}-r\right)+c_{2}\left(y_{n}-r\right)^{2}+c_{3}\left(y_{n}-r\right)^{3}+c_{4}\left(y_{n}-r\right)^{4}+\cdots\right] \\
= & f^{\prime}(r)\left[c_{2} e_{n}^{2}+2\left(c_{3}-c_{2}^{2}\right) e_{n}^{3}+\left(5 c_{2}^{3}+3 c_{4}-7 c_{2} c_{3}\right) e_{n}^{4}+\right.  \tag{9}\\
& \left.+\left(4 c_{5}+24 c_{3} c_{2}^{2}-10 c_{2} c_{4}-6 c_{3}^{2}-12 c_{2}^{4}\right) e_{n}^{5}+\cdots\right]
\end{align*}
$$

and

$$
\begin{gather*}
\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}=c_{2} e_{n}^{2}+2\left(c_{3}-c_{2}^{2}\right) e_{n}^{3}+\left(3 c_{2}^{3}-6 c_{2} c_{3}+3 c_{4}\right) e_{n}^{4}+\cdots  \tag{10}\\
\frac{P_{2}\left(x_{n}, y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}=2 c_{2}+4\left(c_{3}-c_{2}^{2}\right) e_{n}+2\left(4 c_{2}^{3}-7 c_{2} c_{3}+3 c_{4}\right) e_{n}^{2}+\cdots  \tag{11}\\
\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)} \frac{P_{2}\left(x_{n}, y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}=2 c_{2}^{2} e^{2}+8\left(c_{2} c_{3}-c_{2}^{3}\right) e_{n}^{3}+2\left(11 c_{2}^{4}-21 c_{2}^{2} c_{3}+6 c_{2} c_{4}+8 c_{3}^{2}\right) e_{n}^{4}+\cdots \tag{12}
\end{gather*}
$$

Using equations (8), (9) and (12) in (4), we have :

$$
\begin{equation*}
x_{n+1}=r-c_{2} c_{3} e_{n}^{4}+O\left(e_{n}^{5}\right) \tag{13}
\end{equation*}
$$

From (13) and $e_{n+1}=x_{n+1}-r$, we have:

$$
e_{n+1}=-c_{2} c_{3} e_{n}^{4}+O\left(e_{n}^{5}\right)
$$

which shows that (HBM1) has fourth-order convergence.
Theorem 3.2: Let $r$ be a sample zero of sufficient differentiable function $f: \subseteq R \rightarrow$ $R$ for an open interval $I$. If $x_{0}$ is sufficiently close to $r$, then the two-step method defined by (HBM3) has fourth-order convergence.

Proof. Consider to

$$
\begin{gather*}
y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}  \tag{14}\\
x_{n+1}=y_{n}-\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}-\frac{f^{2}(y) P_{2}\left(x_{n}, y_{n}\right)}{2 P_{1}^{3}\left(x_{n}, y_{n}\right)} . \tag{15}
\end{gather*}
$$

Let $r$ be a simple zero of $f$. Since $f$ is sufficiently differentiable, by expanding $f\left(x_{n}\right)$ and $f^{\prime}\left(x_{n}\right)$ about $r$, we get
$f\left(x_{n}\right)=f(r)+\left(x_{n}-r\right) f^{\prime}(r)+\frac{\left(x_{n}-r\right)^{2}}{2!} f^{(2)}(r)+\frac{\left(x_{n}-r\right)^{3}}{3!} f^{(3)}(r)+\frac{\left(x_{n}-r\right)^{4}}{4!} f^{(4)}(r)+\cdots$,
then

$$
\begin{equation*}
f\left(x_{n}\right)=f^{\prime}(r)\left[e_{n}+c_{2} e_{n}^{2}+c_{3} e_{n}^{3}+c_{4} e_{n}^{4}+\cdots\right] \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
f^{\prime}\left(x_{n}\right)=f^{\prime}(r)\left[1+2 c_{2} e_{n}+3 c_{3} e_{n}^{2}+4 c_{4} e_{n}^{3}+5 c_{5} e_{n}^{4}+\cdots\right], \tag{17}
\end{equation*}
$$

where $c_{k}=\frac{1}{k!} \frac{f^{(2)}(r)}{f^{\prime}(r)}, k=1,2,3, \ldots$ and $e_{n}=x_{n}-r$.
Now from (16) and (17), we have

$$
\begin{equation*}
\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=e_{n}-c_{2} e_{n}^{2}+2\left(c_{2}^{2}-c_{3}\right) e_{n}^{3}+\left(7 c_{2} c_{3}-4 c_{2}^{3}+3 c_{4}\right) e_{n}^{4}+\cdots \tag{18}
\end{equation*}
$$

From (14) and (18), we get

$$
\begin{equation*}
y_{n}=r+c_{2} e_{n}^{2}+2\left(c_{3}-c_{2}^{2}\right) e_{n}^{3}+\left(-7 c_{2} c_{3}+4 c_{2}^{3}+3 c_{4}\right) e_{n}^{4}+\cdots \tag{19}
\end{equation*}
$$

From (19), we get,

$$
\begin{align*}
f\left(y_{n}\right)= & f^{\prime}(r)\left[\left(y_{n}-r\right)+c_{2}\left(y_{n}-r\right)^{2}+c_{3}\left(y_{n}-r\right)^{3}+c_{4}\left(y_{n}-r\right)^{4}+\cdots\right] \\
= & f^{\prime}(r)\left[c_{2} e_{n}^{2}+2\left(c_{3}-c_{2}^{2}\right) e_{n}^{3}+\left(5 c_{2}^{3}+3 c_{4}-7 c_{2} c_{3}\right) e_{n}^{4}+\right.  \tag{20}\\
& \left.+\left(4 c_{5}+24 c_{3} c_{2}^{2}-10 c_{2} c_{4}-6 c_{3}^{2}-12 c_{2}^{4}\right) e_{n}^{5}+\cdots\right]
\end{align*}
$$

and

$$
\begin{gather*}
\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}=c_{2} e_{n}^{2}+2\left(c_{3}-c_{2}^{2}\right) e_{n}^{3}+\left(3 c_{2}^{3}-6 c_{2} c_{3}+3 c_{4}\right) e_{n}^{4}+\cdots  \tag{21}\\
{\left[\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}\right]^{2}=c_{2}^{2} e_{n}^{4}+4\left(c_{2} c_{3}-c_{2}^{3}\right) e_{n}^{5}+\cdots}  \tag{22}\\
\frac{P_{2}\left(x_{n}, y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}=2 c_{2}+4\left(c_{3}-c_{2}^{2}\right) e_{n}+2\left(4 c_{2}^{3}-7 c_{2} c_{3}+3 c_{4}\right) e_{n}^{2}+\cdots \tag{23}
\end{gather*}
$$

$$
\begin{equation*}
\frac{1}{2}\left[\frac{f\left(y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}\right]^{2} \frac{P_{2}\left(x_{n}, y_{n}\right)}{P_{1}\left(x_{n}, y_{n}\right)}=c_{2}^{3} e^{4}+6\left(c_{2}^{2} c_{3}-c_{2}^{4}\right) e_{n}^{5}+\cdots \tag{24}
\end{equation*}
$$

combining (19) - (24), we have :

$$
x_{n+1}=r-c_{2} c_{3} e_{n}^{4}+O\left(e_{n}^{5}\right)(25)
$$

From (25), $e_{n+1}=x_{n+1}-r$ and $e_{n}=x_{n}-r$, we have:

$$
e_{n+1}=-c_{2} c_{3} e_{n}^{4}+O\left(e_{n}^{5}\right)
$$

which shows that (HBM3) has fourth-order convergence. In Similar way, we observe that the HBM2, HBM4 and HBM5 have also fourth order convergence as follows

$$
\begin{gathered}
\left.e_{n+1}=c_{2}^{3}-3 c_{4}-c_{2} c_{3}\right) e_{n}^{4}+O\left(e_{n}^{5}\right),(\text { HBM } 2) \\
e_{n+1}=\left(c_{2}^{3}-c_{2}-c_{2} c_{3}\right) e_{n}^{4}+O\left(e_{n}^{5}\right),(\text { HBM } 4) \\
e_{n+1}=\left(3 c_{2}^{3}-c_{2} c_{3}\right) e_{n}^{4}+O\left(e_{n}^{5}\right) .(\text { HBM } 5) .
\end{gathered}
$$

## 4 Numerical examples

For comparisons, we have used the fourth-order Jarratt method [19] (JM) and Ostrowski's method (OM) [11] defined respectively by

$$
\begin{aligned}
& y_{n}=x_{n}-\frac{2}{3} \frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)} \\
& x_{n+1}=x_{n}-\left[1-\frac{3}{2} \frac{f^{\prime}\left(y_{n}\right)-f^{\prime}\left(x_{n}\right)}{3 f^{\prime}\left(y_{n}\right)-f^{\prime}\left(x_{n}\right)}\right] \frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
& y_{n}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)} \\
& x_{n+1}=y_{n}-\frac{f\left(x_{n}\right)}{f\left(x_{n}\right)-2 f\left(y_{n}\right)} \frac{f\left(y_{n}\right)}{f^{\prime}\left(x_{n}\right)} .
\end{aligned}
$$

We consider here some numerical examples to demonstrate the performance of the new modified two-step iterative methods, namely (HBM1) - (HBM5). We compare the classical Newton's method (NM), the predictor-corrector Newton method (PCN), Jarratt method (JM), the Ostrowski's method (OM) and the new modified two-step methods (HBM1) - (HBM5), in this paper. In the Tables 1, 2 the number of iteration is $n=5$ for all our examples. But in Table 1 our examples are tested with precision $\varepsilon=10^{-200}$. The following stopping criteria is used for computer programs: $\left|f\left(x_{n+1}\right)\right|<\varepsilon$.

And the computational order of convergence (COC) can be approximated using the formula,

$$
C O C \approx \frac{\ln \left|\left(x_{n+1}-x_{n}\right) /\left(x_{n}-x_{n-1}\right)\right|}{\ln \left|\left(x_{n}-x_{n-1}\right) /\left(x_{n-1}-x_{n-2}\right)\right|}
$$

Table 1 shows the difference of the root $r$ and the approximation $x_{n}$ to $r$, where $r$ is the exact root computed with 2000 significant digits, but only 25 digits are displayed for $x_{n}$. In Table 1, we listed the number of iterations for various methods. The absolute values of the function $f\left(x_{n}\right)$ and the computational order of convergence (COC) are also shown in Tables 2, 3. All the computations are performed using Maple 15. The following examples are used for numerical testing:

$$
\begin{array}{llcc}
f_{1}(x)=x^{3}+4 x^{2}-10, & x_{0}=1 . & f_{2}(x)=\sin ^{2} x-x^{2}+1, & x_{0}=1.3 . \\
f_{3}(x)=x^{2}-e^{x}-3 x+2, & x_{0}=2 . & f_{4}(x)=\cos x-x, & x_{0}=1.7 . \\
f_{5}(x)=(x-1)^{3}-1, & x_{0}=2.5 . & f_{6}(x)=x^{3}-10, & x_{0}=2 . \\
f_{7}(x)=e^{x^{2}+7 x-30}-1, & x_{0}=3.1 . & &
\end{array}
$$

Results are summarized in Table 1, 2 and Table 3 as it shows, new algorithms are comparable with all of the methods and in most cases gives better or equal results.

Table 1. Comparison of Number of iterations for various methods required such that

| $\left\|f\left(x_{n+1}\right)\right\|<10^{-200}$ |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Method | $f_{1}$ | $f_{2}$ | $f_{3}$ | $f_{4}$ | $f_{5}$ | $f_{6}$ | $f_{7}$ |
| Guess | 1 | 1.3 | 2 | 1.7 | 2.5 | 2 | 3.1 |
| NM | 9 | 8 | 8 | 8 | 9 | 8 | 10 |
| PCN | 5 | 4 | 5 | 5 | 5 | 4 | 5 |
| JM | 5 | 4 | 5 | 5 | 5 | 4 | 5 |
| OM | 5 | 4 | 5 | 5 | 5 | 4 | 5 |
| HBM1 | 4 | 4 | 5 | 5 | 5 | 4 | 6 |
| HBM2 | 5 | 4 | 5 | 5 | 5 | 4 | 5 |
| HBM3 | 4 | 4 | 5 | 5 | 5 | 4 | 5 |
| HBM4 | 5 | 4 | 6 | 5 | 5 | 4 | 5 |
| HBM5 | 5 | 4 | 5 | 5 | 5 | 4 | 5 |

## 5 Conclusions

In numerical analysis, many methods produce sequences of real numbers, for instance the iterative methods for solving nonlinear equations. Sometimes, the convergence of these sequences is slow and their utility in solving practical problems quite limited. Convergence acceleration methods try to transform a slowly converging sequence into a fast convergent one. Due to this, paper has aimed to give a rapidly convergent two-point class for approximating simple roots. As high as possible of convergence order was attained by using as small as possible number of evaluations per full cycle. The local order of our class of iterations was established theoretically, and it has been seen that our class supports the optimality conjecture of Kung-Traub [16]. In the sequel, numerical examples have used in order to show the efficiency and accuracy of the novel methods from our suggested second derivative-free class. Finally, it should be noted that, like all other iterative methods, the new methods from the class (HBM1)-(HBM5) have their own domains of validity and in certain circumstances should not be used.

Table 2. Comparison of different methods

| Method | $x_{0}$ | $x_{5}$ | $C O C$ | $\left\|x_{5}-r\right\|$ | $\left\{\mid f\left(x_{5}\right)\{\mid\right.$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f_{1}$ | 1 |  |  |  |  |
| NM |  | 1.3652300134140968457610286 | 2 | $2.1 \mathrm{e}-9$ | $3.6 \mathrm{e}-19$ |
| PCN |  | 1.3652300134140968457608068 | 3.97 | $1.5 \mathrm{e}-185$ | $7.1 \mathrm{e}-746$ |
| JM | 1.3652300134140968457608068 | 3.98 | $1.5 \mathrm{e}-185$ | $7.1 \mathrm{e}-746$ |  |
| OM | 1.3652300134140968457608068 | 3.99 | $1.4 \mathrm{e}-185$ | $7.1 \mathrm{e}-746$ |  |
| HBM1 | 1.3652300134140968457608068 | 4 | $2.1 \mathrm{e}-236$ | $1 \mathrm{e}-953$ |  |
| HBM2 | 1.3652300134140968457608068 | 3.99 | $1.4 \mathrm{e}-185$ | $7.1 \mathrm{e}-746$ |  |
| HBM3 | 1.3652300134140968457608068 | 4 | $1.2 \mathrm{e}-243$ | $1.3 \mathrm{e}-978$ |  |
| HBM4 | 1.3652300134140968457608068 | 4 | $3.6 \mathrm{e}-123$ | $1.2 \mathrm{e}-495$ |  |
| HBM5 | 1.3652300134140968457608068 | 3.99 | $1.5 \mathrm{e}-125$ | $3.1 \mathrm{e}-505$ |  |
| $f_{2}$ | 1.3 |  |  |  |  |
| NM | 1.4044916482153412260350868 | 2 | $1.5 \mathrm{e}-15$ | $4.8 \mathrm{e}-33$ |  |
| PCN | 1.4044916482153412260350868 | 4 | $3.0 \mathrm{e}-276$ | $8.8 \mathrm{e}-1109$ |  |
| JM | 1.4044916482153412260350868 | 4 | $2.0 \mathrm{e}-277$ | $1.6 \mathrm{e}-1113$ |  |
| OM | 1.4044916482153412260350868 | 4 | $3.0 \mathrm{e}-276$ | $8.8 \mathrm{e}-1109$ |  |
| HBM1 | 1.4044916482153412260350868 | 4 | $5.0 \mathrm{e}-339$ | $1.0 \mathrm{e}-1360$ |  |
| HBM2 | 1.4044916482153412260350868 | 4 | $3.0 \mathrm{e}-276$ | $8.8 \mathrm{e}-1109$ |  |
| HBM3 | 1.4044916482153412260350868 | 4 | $2.3 \mathrm{e}-340$ | $5.1 \mathrm{e}-1366$ |  |
| HBM4 | 1.4044916482153412260350868 | 4 | $7.0 \mathrm{e}-275$ | $2.2 \mathrm{e}-1103$ |  |
| HBM5 | 1.4044916482153412260350868 | 4 | $6.1 \mathrm{e}-226$ | $4.8 \mathrm{e}-907$ |  |
| $f_{3}$ | 2 |  |  |  |  |
| NM | 0.2575302854398607604553673 | 2 | $9.8 \mathrm{e}-12$ | $3.4 \mathrm{e}-25$ |  |
| PCN | 0.2575302854398607604553673 | 3.99 | $2.3 \mathrm{e}-91$ | $5.5 \mathrm{e}-371$ |  |
| JM | 0.2575302854398607604553673 | 4 | $4.1 \mathrm{e}-93$ | $7.0 \mathrm{e}-378$ |  |
| OM | 0.2575302854398607604553673 | 3.99 | $2.3 \mathrm{e}-91$ | $5.5 \mathrm{e}-371$ |  |
| HBM1 | 0.2575302854398607604553673 | 3.99 | $8.0 \mathrm{e}-49$ | $8.3 \mathrm{e}-201$ |  |
| HBM2 | 0.2575302854398607604553673 | 3.99 | $2.3 \mathrm{e}-91$ | $5.5 \mathrm{e}-371$ |  |
| HBM3 | 0.2575302854398607604553673 | 3.99 | $9.5 \mathrm{e}-61$ | $1.6 \mathrm{e}-248$ |  |
| HBM4 | 0.2575302854398607604553673 | 3.95 | $5.1 \mathrm{e}-14$ | $2.6 \mathrm{e}-60$ |  |
| HBM5 | 0.2575302854398607604553673 | 4 | $2.2 \mathrm{e}-156$ | $2.9 \mathrm{e}-231$ |  |
| $f_{4}$ | 1.7 |  |  |  |  |
| NM | 0.7390851332151606416553121 | 1.99 | $2.3 \mathrm{e}-14$ | $2.0 \mathrm{e}-30$ |  |
| PCN | 0.7390851332151606416553121 | 3.99 | $2.6 \mathrm{e}-190$ | $1.9 \mathrm{e}-766$ |  |
| JM | 0.7390851332151606416553121 | 3.99 | $3.5 \mathrm{e}-196$ | $6.1 \mathrm{e}-790$ |  |
| OM | 0.7390851332151606416553121 | 3.99 | $2.6 \mathrm{e}-190$ | $1.9 \mathrm{e}-766$ |  |
| HBM1 | 0.7390851332151606416553121 | 3.99 | $7.2 \mathrm{e}-196$ | $6.6 \mathrm{e}-790$ |  |
| HBM2 | 0.7390851332151606416553121 | 3.99 | $2.6 \mathrm{e}-190$ | $1.9 \mathrm{e}-766$ |  |
| HBM3 | 0.7390851332151606416553121 | 3.99 | $2.7 \mathrm{e}-196$ | $6.7 \mathrm{e}-789$ |  |
| HBM4 | 0.7390851332151606416553121 | 4 | $5.0 \mathrm{e}-111$ | $1.2 \mathrm{e}-448$ |  |
| HBM5 | 0.7390851332151606416553121 | 3.99 | $9.7 \mathrm{e}-184$ | $6.9 \mathrm{e}-740$ |  |
|  |  |  |  |  |  |

Table 3. Comparison of different methods

| Method | $x_{0}$ | $x_{5}$ | $C O C$ | $\left\|x_{5}-r\right\|$ | $\left\|f\left(x_{5}\right)\right\|$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f_{5}$ | 2.5 |  |  |  |  |
| NM | 2.0000000000000113791023781 | 2 | $1.0 \mathrm{e}-5$ | $3.4 \mathrm{e}-12$ |  |
| PCN | 2 | 3.99 | $4.1 \mathrm{e}-121$ | $5.9 \mathrm{e}-488$ |  |
| JM | 2 | 3.99 | $4.1 \mathrm{e}-121$ | $5.9 \mathrm{e}-488$ |  |
| OM | 2 | 3.99 | $4.1 \mathrm{e}-121$ | $5.9 \mathrm{e}-488$ |  |
| HBM1 | 2 | 4 | $1.1 \mathrm{e}-141$ | $1.9 \mathrm{e}-566$ |  |
| HBM2 | 2 | 3.99 | $4.1 \mathrm{e}-121$ | $5.9 \mathrm{e}-488$ |  |
| HBM3 | 2 | 4 | $4.9 \mathrm{e}-154$ | $6.1 \mathrm{e}-620$ |  |
| HBM4 | 2 | 4 | $3.3 \mathrm{e}-175$ | $1.2 \mathrm{e}-704$ |  |
| HBM5 | 2 | 3.99 | $4.7 \mathrm{e}-88$ | $3.9 \mathrm{e}-355$ |  |
| $f_{6}$ | 2 |  |  |  |  |
| NM | 2.1544346900318837217592936 | 2 | $2.2 \mathrm{e}-16$ | $3.2 \mathrm{e}-33$ |  |
| PCN | 2.1544346900318837217592936 | 4 | $1.0 \mathrm{e}-301$ | $1.3 \mathrm{e}-1210$ |  |
| JM | 2.1544346900318837217592936 | 4 | $1.0 \mathrm{e}-301$ | $1.3 \mathrm{e}-1210$ |  |
| OM | 2.1544346900318837217592936 | 4 | $1.0 \mathrm{e}-301$ | $1.3 \mathrm{e}-1210$ |  |
| HBM1 | 2.1544346900318837217592936 | 4 | $1.2 \mathrm{e}-329$ | $1.1 \mathrm{e}-1322$ |  |
| HBM2 | 2.1544346900318837217592936 | 4 | $1.0 \mathrm{e}-301$ | $1.3 \mathrm{e}-1210$ |  |
| HBM3 | 2.1544346900318837217592936 | 4 | $4.2 \mathrm{e}-330$ | $1.4 \mathrm{e}-1324$ |  |
| HBM4 | 2.1544346900318837217592936 | 4 | $3.3 \mathrm{e}-223$ | $7.2 \mathrm{e}-936$ |  |
| HBM5 | 2.1544346900318837217592936 | 4 | $3.2 \mathrm{e}-246$ | $4.0 \mathrm{e}-988$ |  |
| $f_{7}$ | 3.1 |  |  |  |  |
| NM | 3.0000000000899925734814359 | 2.03 | $3.6 \mathrm{e}-4$ | $1.1 \mathrm{e}-7$ |  |
| PCN | 3 | 3.99 | $5.0 \mathrm{e}-101$ | $7.7 \mathrm{e}-405$ |  |
| JM | 3 | 3.99 | $5.2 \mathrm{e}-98$ | $1.0 \mathrm{e}-392$ |  |
| OM | 3 | 3.99 | $5.0 \mathrm{e}-101$ | $7.7 \mathrm{e}-405$ |  |
| HBM1 | 3 | 3.99 | $1.1 \mathrm{e}-46$ | $4.9 \mathrm{e}-187$ |  |
| HBM2 | 3 | 3.99 | $5.0 \mathrm{e}-101$ | $7.7 \mathrm{e}-405$ |  |
| HBM3 | 3 | 3.99 | $1.2 \mathrm{e}-70$ | $6.0 \mathrm{e}-283$ |  |
| HBM4 | 3 | 3.99 | $3.0 \mathrm{e}-102$ | $1.0 \mathrm{e}-409$ |  |
| HBM5 | 3 | 3.99 | $1.0 \mathrm{e}-57$ | $8.7 \mathrm{e}-231$ |  |

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# A Dynamical Analysis of an Autocatalytic Model 

Lakshmi Burra<br>J N T University, College of Engineering, Hyderabad, India<br>e-mail:bslakshmi2000@yahoo.com<br>Uma Maheswari<br>J N T University, College of Engineering, Hyderabad, India<br>e-mail:bslakshmi2000@yahoo.com


#### Abstract

In this paper we study an autocatalytic reaction and we derive the differential equations arising from this reaction. We analyze these equations using phase-space analysis. We next use Center Manifold theory to derive a stable Center Manifold for this system. Since this system is a polynomial differential system, we study the orbits of the system which go or come from infinity using the Poincaré compactification. Interestingly these equations would also represent a population model. All these concepts are illustrated graphically.


Keywords: First order planar systems, Autocatalytic reactions, Center manifold theory, Poincaré Compactification

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## 1 Introduction

The dynamics and chemistry of oscillating reactions has been the subject of study for the last several years, starting with the work of Boris Belousov who stumbled upon an oscillating chemical reaction system. In 1961, ten years after Belousov's initial experiments, new work was initiated by A. M. Zhabotinskii. He quickly reproduced Belousov's results, and soon began working on similar systems. This reaction system, now commonly referred to as the Belousov-Zhabotinskii reaction has been thoroughly studied from both chemical and mathematical perspectives. There are now a large number of both 'real' and 'toy' systems that provide insight into the complex behavior of autocatalytic oscillating systems [7, 9]. Among them are the Lotka-Volterra, the Oregonator and the Brusselator. In this paper we put forward a simple two-gender based population model which is also based on an autocatalytic chemical reaction.

## 2 The Model

We assume that a chemical system is subject to a mass-action kinetics. Considering the kinetics of a reaction, this law states that the rate of an elementary reaction is proportional to the product of the concentrations of the participating molecules. Consider the following sequence of chemical reactions,

$$
\begin{align*}
& A+y \underset{-k_{1}}{\stackrel{k_{1}}{\rightleftharpoons}} x+B  \tag{1a}\\
& x \underset{-k_{2}}{\stackrel{k_{2}}{\rightleftharpoons}} y+2 x  \tag{1b}\\
& x+2 y \underset{-k_{3}}{\stackrel{k_{3}}{\rightleftharpoons}} y  \tag{1c}\\
& C+x \underset{-k_{4}}{\stackrel{k_{4}}{\rightleftharpoons}} y+D  \tag{1d}\\
& E+y \underset{-k_{5}}{\stackrel{k_{5}}{\rightleftharpoons}} y+F \tag{1e}
\end{align*}
$$

Where $k_{1}, k_{2}, k_{3}, k_{4}$ and $k_{5}$ are the rates of the forward reactions, the negative counterparts are the rates of the backward reactions. $x$ and $y$ are the autocatalysts, since they are generated during the sequence of reactions and catalyze the reaction. $A, B, C, D$ are some four chemicals. We denote the concentration of the chemicals by the same symbols as the chemicals themselves. The net result of these equations is

$$
A+C \rightleftharpoons B+D
$$

Let us consider equation (1a). From the Law of Mass Action this implies

$$
\frac{d x}{d t} \propto y
$$

and $(1 b) \Rightarrow$ that

$$
\frac{d x}{d t} \propto-y x^{2}
$$

from which after putting in suitable proportionality constants $(a>0, b>0)$ we have

$$
\frac{d x}{d t}=b y-a y x^{2}
$$

Next from equations (1c) and (1e) we obtain that

$$
\frac{d y}{d t} \propto x y^{2}
$$

and

$$
\frac{d y}{d t} \propto-y
$$

respectively. After putting in suitable proportionality constants we have

$$
\frac{d x}{d t}=-y-b y+a y^{2} x
$$

If $x$ and $y$ denote the composition variables of the two intermediates or the autocatalysts, the rate equations after adjusting suitable proportionality constants take the general form:

$$
\begin{align*}
& \frac{d x}{d t}=f(x, y)  \tag{2}\\
& \frac{d y}{d t}=g(x, y)
\end{align*}
$$

The system we propose for the autocatalysts $x$ and $y$ is

$$
\begin{align*}
& \frac{d x}{d t}=b y-a y x^{2}  \tag{3}\\
& \frac{d y}{d t}=-y-b y+a x y^{2}
\end{align*}
$$

The differential system (3) is an autonomous system as there is no explicit dependence on time $t$. We assume that $f$ and $g$ are continuous and satisfy the Lipschitz condition in a certain bounded domain, D , of the phase space $(x, y)$.

### 2.0.1 A Two-Gender Population Model

The system of equations (3), interestingly could also represent a population model in which one considers two sexes involved in reproduction. (see for example [11] and [3]). In [11], the authors consider a bisexual, non-marriage model which involves a function $H\left[N_{1}(t), N_{2}(t)\right]$ which is taken to be a homogeneous function in $N_{1}(t)$ and $N_{2}(t),\left(N_{1}(t)\right.$ and $N_{2}(t)$ represent the populations of the male and female species respectively) for the sake of scale independence. If we discard the scale independence since, in any case scale-independence is an artificial assumption, in our model this function need not be a homogeneous function in $N_{1}(t)$ and $N_{2}(t)$. This could be a topic of further study.

## 3 Analysis of the system

We now perform an analysis of the system.
The equilibrium points are $(0,0),\left( \pm \sqrt{\frac{b}{a}}, \pm \frac{1+b}{\sqrt{a b}}\right)$.
The last two equilibrium points exist only if $a b>0$ and $b \neq-1$. Since our choice for the parameters $a$ and $b$ is $\geq 0$, all three equilibrium points exist. A quick observation of the system (3) shows that the line $y=0$ is a continuum of equilibrium points. Also it can be seen that the system invariant under the symmetry $(x, y) \rightarrow(-x,-y)$, consequently one needs to study its dynamics only in a half-plane.

Also since the system has a continuum of equilibrium points on the line $y=0$, it is more convenient to study the system without this continuum. We therefore introduce
a change in the independent variable $t \rightarrow \tau$ through $d \tau=y d t$, and we examine the associated system

$$
\begin{align*}
& \frac{d x}{d \tau}=b-a x^{2} \\
& \frac{d y}{d \tau}=-1-b+a x y \tag{4}
\end{align*}
$$

We next linearize the system (4) about the equilibrium points $\left( \pm \sqrt{\frac{b}{a}}, \pm \frac{1+b}{\sqrt{a b}}\right)$, to obtain the coefficient Jacobian Matrix respectively as

$$
\left(\begin{array}{cc}
\mp 2 \sqrt{a b} & 0  \tag{5}\\
\pm \frac{\sqrt{a}(b+1)}{\sqrt{b}} & \pm \sqrt{a b}
\end{array}\right)
$$

Next, we find the Eigenvalues of (5), they are $\{-2 \sqrt{a b}, \sqrt{a b}\}$ and $\{-\sqrt{a b}, 2 \sqrt{a b}\}$ corresponding to the two equilibrium points respectively. Since the Eigenvalues are positive and negative the equilibrium points are saddle points.


Figure 1: The two equilibrium points $\left( \pm \sqrt{\frac{b}{a}}, \pm \frac{1+b}{\sqrt{a b}}\right)$, with $a=18$ and $b=1$.

That the equilibrium points are saddle points can also be seen from the figure Fig. 1, the equilibrium points are marked by the intersection of the dashed lines.

## 4 Absence of closed orbits

We now state the well-known


Figure 2: The rotation of the vector field of the system (4) showing the index of the critical point as -1 .

Bendixson's criterion for the absence of closed orbits. Statement of Bendixson's Criterion: If on a simply connected region $D \subset R^{2}$ the expression (if we consider a system in the form as in equation (2)), $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}$ is not identically zero and does not change sign, then equation (2) has no closed orbits lying entirely in $D$.

In order to apply Bendixson's Criterion consider the system (3) with $f(x, y)=b y-$ $a y x^{2}$ and $g(x, y)=-y-b y+a x y^{2}$, then the quantity $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}=-b-1$.

In order to satisfy Bendixson's Criterion, $b \neq 1$ and $b \geq 0$. For $b$ satisfying these conditions the system has no closed orbits. Since we choose in any case that $b \neq 1$ and $b \geq 0$ we can say that the system under consideration has no closed orbits.
Topological Index or Poincaré Index A fundamental concept in vector field topology is the so-called Poincaré index of a simple closed curve: It measures the number of rotations of the vector field while traveling along the curve in a positive direction [8]. The index of a critical point is the index of a simple closed curve around the critical point enclosing no other singular point. Mathematically this is calculated for a closed curve $\gamma$ by the following integral index ${ }_{\gamma}=\frac{1}{2 \pi} \oint_{\gamma} d \phi$ where $\phi$ is the angle the vector field traverses around the curve $\gamma$ in an anti-clockwise direction.

From Fig. 2 it can be seen that the vector field makes one complete rotation in the clockwise direction around a simple closed curve surrounding the equilibrium point. Hence the index of the critical point is -1 .

This also shows that there are no closed orbits surrounding the equilibrium point. By symmetry the second equilibrium point can also be seen to have index -1 .

## 5 Poincaré Compactification

In order to study the behavior of the trajectories of a planar polynomial differential system

$$
\begin{align*}
\dot{x} & =\mathbf{P}(x, y) \\
\dot{y} & =\mathbf{Q}(x, y) \tag{6}
\end{align*}
$$

near infinity a compactification is generally used. We use the the Poincare compactification $[4,5]$. The Poincaré compactification relies on stereographic projection of the sphere onto the plane, for studying the behavior of trajectories near infinity making use the so called Poincaré sphere, introduced by Poincaré [6]. This has the advantage that the singular points at infinity are spread out along the equator of the sphere. The Poincaré compactification, enables one to draw the trajectories in a finite region and controls the orbits which tend to or come from infinity.

Poincaré compactification works as follows: Firstly we consider $R^{2}$ as a plane in $R^{3}$ defined by $\left(y_{1}, y_{2}, y_{3}\right)=(x, y, 1)$. Next, we consider the sphere $\mathbb{S}^{2}=y \in R^{3}: y_{1}^{2}+y_{2}^{2}+y_{3}^{2}=1$ which we will call here the Poincaré sphere; This sphere is tangent to $R^{2}$ at the point ( 0 , 0,1 ). We may divide this sphere into $H_{+}=y \in \mathbb{S}^{2}: y^{3}>0$ (the northern hemisphere), $H_{-}=\left\{y \in \mathbb{S}^{2}: y^{3}<0\right\}$ (the southern hemisphere) and $\mathbb{S}^{1}=\left\{y \in \mathbb{S}^{2}: y_{3}=0\right\}$ (the equator).

We consider the projection of the vector field $X$ from $R^{2}$ to $\mathbb{S}^{2}$ given by the projections $f^{+}: R^{2} \rightarrow \mathbb{S}^{2}$ and $f^{-}: R^{2} \rightarrow \mathbb{S}^{2}$. Or $f^{+}(x)$ (respectively, $f^{-}(x)$ ) is the intersection of the straight line passing through the point y and the origin with the northern (respectively, southern) hemisphere of $\mathbb{S}^{2}$.
$f^{+}(x)=\left(\frac{x}{\Delta}, \frac{y}{\Delta}, \frac{1}{\Delta},\right), f^{-}(x)=\left(-\frac{x}{\Delta},-\frac{y}{\Delta},-\frac{1}{\Delta},\right)$ where $\Delta(x)=\sqrt{x^{2}+y^{2}+1}$.
We thus obtain induced vector fields in each hemisphere. The induced vector field on $H_{+}$is $\bar{X}(y)=D f^{+}(x) X(x)$, where $y=f^{+}(x)$, and the one in $H_{-}$is $\bar{X}(y)=D f^{-}(x) X(x)$, where $y=f^{-}(x)$ where $D X$ represents the linear part of the vector field $X$.

As is usual in working with curved surfaces, we use charts or planes for calculational purposes. For $\mathbb{S}^{2}$ we use the six local planes given by $U_{k}=\left\{y \in \mathbb{S}^{2}: y_{k}>0\right\}, W_{k}=\{y \in$ $\left.\mathbb{S}^{2}: y_{k}<0\right\}$ for $k=1,2,3$. The corresponding local maps $\phi_{k}: U_{k} \rightarrow R^{2}$ and $\psi_{k}: W_{k} \rightarrow R^{2}$ are defined as $\left(y_{m} / y_{k}, y_{n} / y_{k}\right)$ for $m<n$ and $m, n \neq k$. We denote by $z=(u, w)$ the value of $\phi_{k}(y)$ or $\psi_{k}(y)$ for any $k$, such that $(u, w)$ will take on different values depending on the plane we are considering. The points of $\mathbb{S}^{1}$ in any chart have $w=0$.

With this preliminary notation it can be deduced (see [2]) that on $U_{1}$,for example $(u, w)=\left(\frac{y}{x}, \frac{1}{x}\right)$ and for system (6) we have

$$
\begin{align*}
\dot{u} & =w^{d}\left[-u \mathbf{P}\left(\frac{1}{w}, \frac{u}{w}\right)+\mathbf{Q}\left(\frac{1}{w}, \frac{u}{w}\right)\right] \\
\dot{w} & =-w^{d+1} \mathbf{P}\left(\frac{1}{w}, \frac{u}{w}\right) \tag{7}
\end{align*}
$$

Where $d$ is the maximum of the degree of the polynomial fields $\mathbf{P}$ or $\mathbf{Q}$. On the plane $U_{2}$,

$$
\begin{align*}
\dot{u} & =w^{d}\left[u \mathbf{P}\left(\frac{u}{w}, \frac{1}{w}\right)-u \mathbf{Q}\left(\frac{u}{w}, \frac{1}{w}\right)\right] \\
\dot{w} & =-w^{d+1} \mathbf{Q}\left(\frac{u}{w}, \frac{1}{w}\right) \tag{8}
\end{align*}
$$

On the plane $U_{3}$ it is

$$
\begin{align*}
\dot{u} & =\mathbf{P}(u, w)  \tag{9}\\
\dot{w} & =\mathbf{Q}(u, w) . \tag{10}
\end{align*}
$$

For the other three planes $W_{i}, i=1,2,3$, the expression is the same as for the $U_{i}^{\prime} s$ multiplied by $(-1)^{d-1}$, for $i=1,2,3$.

With these equations we evaluate $U_{1}, U_{2}, W_{1}, W_{2}$, for the system of equations (3) with $a=1$ and $b=1$, to obtain its Poincaré Compactification.

Using ( $x=1 / z_{2}, y=z_{1} / z_{2}$ ): The differential system on the $U_{1}$ chart is:

$$
\begin{aligned}
& z_{1}^{\prime}=-z_{1} z_{2}^{2}+2 z_{1}-2 z_{2}^{2} \\
& z_{2}^{\prime}=-z_{2}^{3}+z_{2}
\end{aligned}
$$

$(0,0)$ is an unstable node here. The differential system on the $W_{1}$ chart is:

$$
\begin{aligned}
& z_{1}^{\prime}=z_{1} z_{2}^{2}-2 z_{1}-2 z_{2}^{2} \\
& z_{2}^{\prime}=z_{2}^{3}-z_{2}
\end{aligned}
$$

The differential system on the $U_{2}$ chart is:

$$
\begin{aligned}
& z_{1}^{\prime}=z_{2}^{2}-2 z_{1}^{2}+2 z_{1} z_{2}^{2} \\
& z_{2}^{\prime}=2 z_{2}^{3}-z_{2} z_{1}
\end{aligned}
$$

Here $(0,0)$ is nonelementary .
The differential system on the $W_{2}$ chart is:

$$
\begin{aligned}
& z_{1}^{\prime}=z_{2}^{2}-2 z_{1}^{2}-2 z_{1} z_{2}^{2} \\
& z_{2}^{\prime}=-2 z_{2}^{3}-z_{2} z_{1}
\end{aligned}
$$

Using the $P_{4}$ software described in [2] we plot the phase portrait on the Poincaré disk. Fig. 4 shows the stable and unstable separatrices on the Poincaré disk along with two saddle points. Two other equilibrium points, one a stable focus and the other an unstable focus are located on the equator of $\mathbb{S}^{2}$ at $\mathbb{S}^{2}$. These can be seen in Fig. 4.

## 6 Existence of a Center Manifold

We now try to see if there is any region in the phase-space in which the solutions are invariant. To this end we set about finding an invariant center-manifold. But first we need a few technical preliminaries: We consider vector fields of the form

$$
\begin{align*}
\dot{x} & =A x+\hat{f}(x, y), \quad(x, y) \in R^{c} \times R^{s}, \\
\dot{y} & =B y+\hat{g}(x, y) \tag{11}
\end{align*}
$$

where $\hat{f}(0,0)=0, D \hat{f}(0,0)=0, \hat{g}(0,0)=0, D \hat{g}(0,0)=0$. In the above, $A$ is a $c \times c$ matrix having eigenvalues with zero real parts, $B$ is an $s \times s$ matrix having eigenvalues with negative real parts, and $\hat{f}$ and $\hat{g}$ are $C^{r}$ functions $(r \geq 2)$.

Definition 6.1 [10] (Center Manifold) An invariant manifold will be called a center manifold for (11) if it can locally be represented as follows

$$
W^{c}(0)=\left\{(x, y) \in R^{c} \times R^{s}|y=h(x),|x|<\delta, h(0)=0, D h(0)=0\}\right.
$$

for $\delta$ sufficiently small.


Figure 3: The phase portrait on the Poincaré disk


Figure 4: The Separatrices and the Equilibrium points on the Poincaré disk

Theorem 6.1 (Existence) There exists a $C^{r}$ center manifold for (11). The dynamics of (11) restricted to the center manifold is, for $u$ sufficiently small, given by the following c-dimensional vector field,
$\dot{u}=A u+\hat{f}(u, h(u)), u \in R^{c}$.

The system (3) can be re-formulated in terms of the functions $f(x, y)$ and $g(x, y)$ and $A$ and $B$ with

$$
A=0 ; B=-1 ; \hat{f}(x, y)=b y-a y x^{2} ; \hat{g}(x, y)=-b y+a x y^{2}
$$

Then clearly the conditions for the existence of a center manifold are satisfied. In the above, $A$ is a $c \times c$ matrix having eigenvalues with zero real parts, in our example $A$ is a scalar equal to $0, B$ is an $s \times s$ matrix having eigenvalues with negative real parts, in this case $B$ is a negative scalar $=-1$, and $\hat{f}$ and $\hat{g}$ are $C^{r}$ functions $(r \geq 2)$. Then by Theorem (6.1), there exists a center manifold to be obtained from the equation
$\dot{u}=A u+\hat{f}(u, h(u))$, that is $\dot{u}=0+\hat{f}(u, h(u))$. We obtain $h(u)$ using a method put forth in [10]. We outline the procedure briefly to obtain the function $h(x)$. We derive an equation that $h(x)$ (or $h(u)$ ) must satisfy in order for its graph to be a center manifold for (3). Towards this end let us assume that we have a center manifold

$$
W^{2}(0)=\left\{(x, y) \in R^{2} \times R^{2}|y=h(x),|x|<\delta, h(0)=0, D h(0)=0\}\right.
$$

Let $h(x)=c x+d x^{2}+s x^{3} \ldots$, where $c$ and $d$ and $s$ are constants to be determined.
Starting with the assumption of invariance of $W^{2}(0)$ under the dynamics of (3), we derive a quasilinear partial differential equation that $h(x)$ satisfies. This is derived in the following manner: The $(x, y)$ coordinates of any point on the center manifold $W^{2}(0)$ must satisfy the function

$$
\begin{equation*}
y=h(x) \tag{12}
\end{equation*}
$$

If we differentiate Equation (12) with respect to $t$, we obtain

$$
\begin{equation*}
\dot{y}=\frac{h(x)}{d t} \dot{x} \tag{13}
\end{equation*}
$$

Since any point on $W^{2}(0)$ satisfies the dynamics of $(3)$, so $(\dot{x}, \dot{y})$ from (3) should satisfy (13). In general equation (11) could then be written as

$$
\begin{align*}
\dot{x} & =A x+\hat{f}(x, h(x)) \\
\dot{y} & =B h(x)+\hat{g}(x, h(x)) \tag{14}
\end{align*}
$$

Then equation (11) becomes

$$
\begin{equation*}
\mathcal{N} \equiv D h(x)[A x+\hat{f}(x, h(x))]-B h(x)+\hat{g}(x, h(x)) \tag{15}
\end{equation*}
$$

If equation (15) is solved we obtain the invariant center manifold $h(x)$.

## 7 Computation of the Center Manifold

Consider (3), for the purpose of computing the center manifold we set the functions $\hat{f}(x, y)$, and $\hat{g}(x, y)$ from equation (11) as

$$
\begin{aligned}
\hat{f}(x, y) & \equiv b y-a y x^{2} \\
\hat{g}(x, y) & \equiv-b y+a x y^{2}
\end{aligned}
$$

constants $A=0$ and $B=-1$. Let $h(x)=c x+d x^{2}+s x^{3} \ldots$ Then the system satisfies the conditions of Theorem (6.1) required for the existence of a center manifold. Substituting for $h(x)$ we have

$$
\begin{align*}
& D h(x)[A x+\hat{f}(x, h(x))]-B h(x)+\hat{g}(x, h(x)) \\
\equiv & \left(c+2 d x+3 s x^{2}\right)\left[x\left(b-a x^{2}\right)(c+x(d+s x))\right] \\
& -(-1)\left(c x+d x^{2}+s x^{3} \ldots\right)+a x^{3}(c+x(d+s x))^{2} \\
= & x^{3}\left(-2 a c^{2}+4 b c s+2 b d^{2}+(b+1) s\right)+x \\
& \left(b c^{2}+(b+1) c\right)+x^{2}(3 b c d+(b+1) d)+\ldots \tag{16}
\end{align*}
$$

Equating coefficients of the powers of $x$ of equation (16) to zero, we obtain $s=-\frac{2 a(b+1)}{3 b^{2}}, c=$ $\frac{-b-1}{b}$ and $d=0$ and hence $h(x)=\frac{-b-1}{b} x-\frac{2 a(b+1)}{3 b^{2}} x^{3}$.

Fig. 5 show the center manifold as a thick line along with the level lines of the system for $a=18, b=1$.


Figure 5: The center manifold shown as the thick line, with $a=18$ and $b=1$.

## 8 Conclusion

We formulated a model from an autocatalytic reaction, which thus represents the chemical reaction. We hypothesized that this could also represent a population model with two genders. Initially we performed a simple phase plane analysis, and a study of its equilibrium points. Since the system is a planar polynomial system, we studied its Poincaré compactification to understand its equilibrium points at infinity and the separatrices.

The theory that we developed so far tells us that equation (3) has an invariant manifold $y=h(x)$, which we have derived and plotted against the phase portrait of the system in Fig. 5. While the equilibrium points are saddle points, we have an invariant manifold which, and as is well-known is not necessarily unique [1].

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# Chaos in the Planar Two-Body Coulomb Problem with a Uniform Magnetic Field 

Vladimir Zhdankin<br>Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, USA<br>e-mail: zhdankin@wisc.edu<br>J. C. Sprott<br>Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, USA<br>email: sprott@physics.wisc.edu


#### Abstract

The dynamics of the classical two-body Coulomb problem in a uniform magnetic field are explored numerically in order to determine when chaos can occur. The analysis is restricted to the configuration of planar particles with an orthogonal magnetic field, for which there is a four-dimensional phase space. Parameters of mass and charge are chosen to represent physically motivated systems. To check for chaos, the largest Lyapunov exponent and Poincaré section are determined for each case. We find chaotic solutions when particles have equal signs of charge. We find cases with opposite signs of charge to be numerically unstable, but a Poincaré section shows that chaos occurs in at least one case.


Keywords: low-dimensional chaos, non-linear dynamics, Hamiltonian systems.

## 1 Introduction

Chaotic systems are bounded dynamical systems that exhibit a sensitive dependence to initial conditions. Generally, they are aperiodic and have governing equations that are nonlinear [1]. There is an interest in studying chaos in systems with a small number of degrees of freedom, such as the three-body gravitational problem [2]. The presence of chaos in such simple systems suggests that it is a fundamental feature of nature. By studying the simplest chaotic systems, we can better understand how chaos arises.

The two-body Coulomb problem in a uniform magnetic field is one of the simplest classical systems that can exhibit chaos. This is particularly true for the two-dimensional case, where the charged particles undergo planar motion and the magnetic field is directed orthogonal to the plane. If the system is simplified any further, for example by removing the magnetic field or one particle, then the equations become integrable. Thus the problem is interesting for its simplicity.

The two-body Coulomb problem in a uniform magnetic field is also applicable to a number of physical situations. It is commonly studied in the context of the classical hydrogen atom in a magnetic field [3]. Although quantum mechanics is essential for making accurate predictions for systems on the atomic scale, the classical model can approximate high energy states and be useful in trying to understand quantum chaos [4]. The problem may also be relevant for describing the interactions of ions in a strong magnetic field. This is appropriate, for example, in the magnetic fields found near white dwarfs and neutron stars, where the properties of matter are drastically modified [5, $6,7]$. In this case, the magnetic field is approximately uniform at the level of particle interactions. Information about the microscopic interactions in such systems could lead to observable global consequences, similar to what has been found in large populations of coupled oscillators [8].

The problem has been studied to various extents by both analytical and numerical means. On the analytical side, Curilef and Claro obtained solutions for the twodimensional problem in the special case where particles have equal mass and equal magnitude of charge [9]. In addition, Pinheiro and MacKay have performed a mathematically rigorous analysis of the general problem in a recent series of two papers [10, 11]. For the two-dimensional problem, they found that all solutions are bounded in space and that the special case of equal gyrofrequencies $\left(q_{1} / m_{1}=q_{2} / m_{2}\right)$ is integrable. Furthermore, they inferred that chaos exists in cases with opposite signs of charge unless gyrofrequencies sum to zero. However, they did not establish what happens when the gyrofrequencies sum to zero and whether there is chaos for cases with equal signs of charge. Due to the complicated nature of the general solutions, the problem is well suited for numerical analysis. Previous numerical studies have been performed primarily for the special case of the classical hydrogen atom in three-dimensional space, which was found to be chaotic by Schmelcher and Cederbaum [12], as well as Friedrich and Wintgen [3]. However, numerical studies of the problem for other sets of particles are scant or nonexistent.

The goal of this paper is to investigate the question of what dynamics are possible in the general solution of the two-dimensional problem. Since the special case in which charges sum to zero (which includes proton-electron, positron-electron) has been studied by others $[9,10,12,3]$, we consider cases in which the two charges do not sum to zero. The charges and masses are chosen to represent cases of physical significance, which reduces the size of the parameter space to be explored. The equations are then solved numerically and two methods are used to test for chaos. The first is computation of the largest Lyapunov exponent, which measures the rate at which nearby trajectories diverge and is positive $(\lambda>0)$ for chaotic solutions. The second is construction of the Poincaré section, which exhibits a chaotic sea for chaotic solutions. As a result of our numerical analysis, we discover many chaotic solutions when charges have equal signs. On the other hand, we find that cases with opposite signs of charge are numerically unstable, which prevents
the Lyapunov exponent from converging accurately. However, a Poincaré section suggests that chaos does occur in at least one of these cases. We also investigate the special case in which gyrofrequencies sum to zero, but are unable to find any chaotic solutions.

## 2 Equations of motion

In this section, the equations of motion for the problem are presented. First, consider the problem in Cartesian coordinates, where two charged particles confined to the $x-y$ plane interact via the Coulomb force in the presence of a uniform magnetic field oriented in the $z$ direction. Then there is an eight-dimensional phase space, which can be written in terms of the positions and kinetic momenta of the particles $\left(x_{1}, y_{1}, p_{x 1}, p_{y 1}, x_{2}, y_{2}, p_{x 2}\right.$, $p_{y 2}$ ). The equations of motion read

$$
\begin{align*}
\dot{p}_{x 1} & =\frac{k_{e} q_{1} q_{2}\left(x_{1}-x_{2}\right)}{\left[\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}\right]^{3 / 2}}+\frac{q_{1} B}{m_{1}} p_{y 1} \\
\dot{p}_{y 1} & =\frac{k_{e} q_{1} q_{2}\left(y_{1}-y_{2}\right)}{\left[\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}\right]^{3 / 2}}-\frac{q_{1} B}{m_{1}} p_{x 1} \\
\dot{x}_{1} & =\frac{1}{m_{1}} p_{x 1} \\
\dot{y}_{1} & =\frac{1}{m_{1}} p_{y 1} \\
\dot{p}_{x 2} & =\frac{k_{e} q_{1} q_{2}\left(x_{2}-x_{1}\right)}{\left[\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}\right]^{3 / 2}}+\frac{q_{2} B}{m_{2}} p_{y 2} \\
\dot{p}_{y 2} & =\frac{k_{e} q_{1} q_{2}\left(y_{2}-y_{1}\right)}{\left[\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}\right]^{3 / 2}}-\frac{q_{2} B}{m_{2}} p_{x 2} \\
\dot{x}_{2} & =\frac{1}{m_{2}} p_{x 2} \\
\dot{y}_{2} & =\frac{1}{m_{2}} p_{y 2} \tag{1}
\end{align*}
$$

where $B$ is the magnetic field strength, $m_{1}$ and $m_{2}$ are the masses of the particles, $q_{1}$ and $q_{2}$ are the electrical charges of the particles, and $k_{e}$ is Coulomb's constant. There are four conserved quantities: the energy $E$, the $x$-component of linear momentum $P_{x}$, the $y$-component of linear momentum $P_{y}$, and angular momentum $L$. These quantities are given by

$$
\begin{align*}
E= & \frac{p_{1 x}^{2}+p_{1 y}^{2}}{2 m_{1}}+\frac{p_{2 x}^{2}+p_{2 y}^{2}}{2 m_{2}}-\frac{k_{e} q_{1} q_{2}}{\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}}} \\
P_{x}= & p_{1 x}+p_{2 x}-q_{1} B y_{1}-q_{2} B y_{2} \\
P_{y}= & p_{1 y}+p_{2 y}+q_{1} B x_{1}+q_{2} B x_{2}  \tag{2}\\
L= & \left(x_{1} p_{1 y}-y_{1} p_{1 x}\right)+\left(x_{2} p_{2 y}-y_{2} p_{2 x}\right) \\
& +\frac{1}{2} B q_{1}\left(x_{1}^{2}+y_{1}^{2}\right)+\frac{1}{2} B q_{2}\left(x_{2}^{2}+y_{2}^{2}\right)
\end{align*}
$$

Although the full equations of motion (Eq. 1) can be solved numerically, it is preferable to use equations in a reduced phase space. Coordinate transformations derived by Pinheiro and MacKay [10] use the conservation laws to reduce the number of phase space dimensions from eight to four. The following transformations will require that charges have a nonzero sum, $q_{1}+q_{2} \neq 0$. The case for $q_{1}+q_{2}=0$ must be treated separately because the center of mass undergoes a drift, and that transformation will not be considered here. The reduced phase space consists of the variables $r, p_{r}, \phi$, and $p_{\phi}$, defined by

$$
\begin{align*}
r & =\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}} \\
p_{r} & =\frac{\left[p_{1 x}-(q-1) p_{2 x}\right]\left(x_{1}-x_{2}\right)+\left[p_{1 y}-(q-1) p_{2 y}\right]\left(y_{1}-y_{2}\right)}{q r} \\
\phi & =\frac{1}{2 \mu} \arctan \left(\frac{\left(p_{1 x}+p_{2 x}\right)\left(x_{1}-x_{2}\right)+\left(p_{1 y}+p_{2 y}\right)\left(y_{1}-y_{2}\right)}{\left(p_{1 y}+p_{2 y}\right)\left(x_{1}-x_{2}\right)-\left(p_{1 x}+p_{2 x}\right)\left(y_{1}-y_{2}\right)}\right) \\
p_{\phi} & =\left(p_{1 x}+p_{2 x}\right)^{2}+\left(p_{1 y}+p_{2 y}\right)^{2} \tag{3}
\end{align*}
$$

and the new parameters are defined as

$$
\begin{align*}
m & =\frac{1+m_{2}}{m_{2}} \\
q & =\frac{1+q_{2}}{q_{2}} \\
\mu & =B\left(1+q_{2}\right) \\
p_{\theta} & =-2 B\left(1+q_{2}\right) L+P_{x}^{2}+P_{y}^{2} \\
\epsilon & =\frac{q_{2} B}{1+q_{2}}=\frac{q-1}{q^{2}} \mu \tag{4}
\end{align*}
$$

where units have been chosen such that $m_{1}=1$ and $q_{1}=1$. It is clear that $r>0$ and $p_{\phi}>0$, with singularities located at $r=0$ and at $p_{\phi}=0$. Also note that $\phi$ is periodic on the interval $\left(0, \frac{\pi}{\mu}\right)$.

The equations of motion can obtained by differentiating Eq. 3 with respect to time and substituting the Cartesian equations of motion (Eq. 1), or alternatively they can be derived from the Hamiltonian (representing conserved energy),

$$
\begin{align*}
H= & \frac{m}{2} p_{r}^{2}+\frac{m}{8} \frac{\left(p_{\theta}-p_{\phi}\right)^{2}}{\mu^{2} r^{2}}+\frac{m \epsilon^{2}}{8} r^{2}+\frac{m \epsilon}{4 \mu}\left(p_{\theta}+p_{\phi}\right)+\left(1-\frac{m}{q}\right)\left(\frac{q-2}{2 q}\right) p_{\phi} \\
& +\frac{k_{e}}{(q-1) r}+\left(1-\frac{m}{q}\right)\left[p_{r} \sin (2 \mu \phi)-\frac{1}{2}\left(\epsilon r+\frac{p_{\theta}-p_{\phi}}{\mu r}\right) \cos (2 \mu \phi)\right] p_{\phi}^{1 / 2} \tag{5}
\end{align*}
$$

by using Hamilton's equations for each component of momentum and position,

$$
\begin{equation*}
\dot{x}_{i}=\frac{\partial H}{\partial p_{i}}, \dot{p}_{i}=-\frac{\partial H}{\partial x_{i}} \tag{6}
\end{equation*}
$$

The equations of motion in the reduced coordinates are then

$$
\begin{align*}
\dot{r}= & m p_{r}+\left(1-\frac{m}{q}\right) \sin (2 \mu \phi) p_{\phi}^{1 / 2} \\
\dot{p}_{r}= & \frac{m}{4} \frac{\left(p_{\theta}-p_{\phi}\right)^{2}}{\mu^{2} r^{3}}-\frac{m \epsilon^{2}}{4} r+\frac{k_{e}}{(q-1) r^{2}}+\frac{1}{2}\left(1-\frac{m}{q}\right)\left(\epsilon-\frac{p_{\theta}-p_{\phi}}{\mu r^{2}}\right) \cos (2 \mu \phi) p_{\phi}^{1 / 2} \\
\dot{\phi}= & \left(1-\frac{m}{q}\right)\left(\frac{q-2}{2 q}\right)+\frac{m}{4}\left(\frac{\epsilon}{\mu}-\frac{p_{\theta}-p_{\phi}}{\mu^{2} r^{2}}\right) \\
& +\frac{1}{2}\left(1-\frac{m}{q}\right)\left[p_{r} \sin (2 \mu \phi)-\frac{1}{2}\left(\epsilon r+\frac{p_{\theta}-3 p_{\phi}}{\mu r}\right) \cos (2 \mu \phi)\right] p_{\phi}^{-1 / 2} \\
\dot{p}_{\phi}= & -2 \mu\left(1-\frac{m}{q}\right)\left[p_{r} \cos (2 \mu \phi)+\frac{1}{2}\left(\epsilon r+\frac{p_{\theta}-p_{\phi}}{\mu r}\right) \sin (2 \mu \phi)\right] p_{\phi}^{1 / 2} \tag{7}
\end{align*}
$$

The five parameters are the relative mass $m(\geq 1)$, the relative charge $q$, the strength of the magnetic field $\mu$, the strength of the Coulomb force $k_{e}(\geq 0)$, and the initial total momentum $p_{\theta}$. If the first particle is taken to be the less massive of the pair, then $1 \leq m \leq 2$.

To obtain numerical solutions, a fourth-order Runge-Kutta algorithm with an adaptive step size is used. We prefer this method over a symplectic integrator because it is much simpler to implement for the given equations of motion. The solutions are independently confirmed using MATLAB and PowerBASIC. The accuracy in each case is primarily checked by monitoring the energy. As numerical error accumulates, the energy given by Eq. 5 drifts away from the initial energy. Since energy is conserved in the actual solution, the energy drift is a signature of numerical error. Typically, we demand that the energy stays constant to six significant digits throughout the simulation.

## 3 Results

The parameters of $m$ and $q$ are chosen from the physically motivated cases in Table 1. The table also shows whether any chaotic solutions were found. These cases cover a range of parameter values, although there are other combinations of particles that may be interesting but were not considered here.

The particles shown in Table 1 are the proton $p$, electron $e$, deuteron $d$, triton $t$ (tritium nucleus), helion $h$ (helium- 3 nucleus), and alpha particle $\alpha$ (helium- 4 nucleus). Additionally, the antiparticles $\bar{p}$ and $\bar{d}$ are included in a case. The case of $\bar{d}-\alpha$ is very unlikely to occur in nature, but is interesting because of the fact that the gyrofrequencies sum to zero, for which Pinheiro and MacKay were unable to establish what happens [10].

Out of the cases listed in Table 1, four were found to exhibit chaos. Parameters and initial conditions that give a chaotic solution for these cases are listed in Table 2. An estimation of the largest Lyapunov exponent is also given. An example of possible trajectories in Cartesian coordinates is shown in Fig. 1.

The largest Lyapunov exponent could not be computed accurately for $\bar{p}-\alpha$. This is due to numerical error (as deduced from energy drift) that accumulates due to close approaches of the particles. Since the Coulomb force in this case is attractive, the particles can come arbitrarily close to the singularity at $r=0$, and this causes problems over long

Table 1: Some possible two-body problems with $q_{1}+q_{2} \neq 0$

| 1 | 2 | $m_{2} / m_{1}$ | $q_{2} / q_{1}$ | $m$ | $q$ | Chaotic? |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $p$ | $p$ | 1 | 1 | 2 | 2 | No, integrable |
| $p$ | $d$ | 2 | 1 | 1.5 | 2 | No |
| $p$ | $h$ | 3 | 2 | $4 / 3$ | 1.5 | Yes |
| $p$ | $t$ | 3 | 1 | $4 / 3$ | 2 | Yes |
| $p$ | $\alpha$ | 4 | 2 | 1.25 | 1.5 | No |
| $d$ | $t$ | 1.5 | 1 | $5 / 3$ | 2 | Yes |
| $d$ | $\alpha$ | 2 | 2 | 1.5 | 1.5 | No, integrable |
| $e$ | $\alpha$ | 8000 | -2 | 1.000125 | 0.5 | No |
| $\bar{p}$ | $\alpha$ | 4 | -2 | 1.25 | 0.5 | Yes |
| $\bar{d}$ | $\alpha$ | 2 | -2 | 1.5 | 0.5 | No |



Figure 1: An example of possible trajectories for the two-body Coulomb problem in a uniform magnetic field. The parameters are taken from the chaotic case of a deuteron (blue) and triton (red).
simulations. This prevents computation of the Lyapunov exponent since averaging over a long simulation time is required, but it is still possible to construct a Poincaré section by collecting points until the error becomes too large.

Poincaré sections are constructed by plotting ( $r, p_{r}, p_{\phi}$ ) when $\phi$ crosses a chosen value. This is performed for a set of representative initial conditions. To further reduce the dimensionality of the Poincaré section, the initial conditions are chosen to all have equal energies. This restricts the Poincaré section to a two-dimensional surface of constant energy. Chaotic solutions fill a region of the Poincaré section known as the chaotic sea, while quasiperiodic solutions show up as closed curves.

The surface of constant energy for $p-h$ is ellipsoidal, so it is possible to show the Poincaré section as two projections on the $r-p_{r}$ plane. These projections are shown in Fig. 2. The Poincaré section predominantly consists of a chaotic sea, with some islands of quasiperiodicity.

The surface of constant energy has a more complicated topology for the cases of $p-t$


Figure 2: Poincaré section for the two-body system of a helion and proton taken at $\phi=$ $\pi / 2$. The surface of constant energy can be conveniently separated into two projections onto the $r$ - $p_{r}$ plane.


Figure 3: The projected Poincaré section for the two-body system of a proton and triton taken at $\phi=0$. The color of each point indicates the $p_{\phi}$ position.

Table 2: Parameters and initial conditions for chaotic cases

| 1 | 2 | $m$ | $q$ | $\mu$ | $p_{\theta}$ | $k_{e}$ | $\left[r, p_{r}, \phi, p_{\phi}\right]_{t=0}$ | Remarks |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h$ | $p$ | 4 | 3 | -1 | 1 | 1 | $[2.8,0,0,0.9370]$ | $\lambda \approx 0.0155$ |
| $t$ | $p$ | 4 | 2 | -1 | 2 | 1 | $[0.4,0,0,3.1109]$ | $\lambda \sim 0.02$ |
| $d$ | $t$ | $5 / 3$ | 2 | 1 | 3 | 1 | $[4.5,0,0,4.8038]$ | $\lambda \approx 0.0175$ |
| $\bar{p}$ | $\alpha$ | 1.25 | 0.5 | -0.5 | 1 | 3 | $[3.0,0,0,1.4102]$ | Numerically unstable |



Figure 4: The projected Poincaré section for the two-body system of a deuteron and triton taken at $\phi=\frac{\pi}{2|\mu|}$. The color of each point indicates the $p_{\phi}$ position.
and $d-t$. In these cases, the projection of the Poincaré section onto the $r-p_{r}$ plane is taken, and then a coloring scheme is used to represent the $p_{\phi}$ value at each point. The resulting Poincaré sections are shown in Fig. 3 and Fig. 4. The case of $p-t$ has a phase space with a roughly equal amount of chaotic and quasiperiodic regions, while the case of $d-t$ almost exclusively consists of a chaotic sea.

The Poincaré section for $\bar{p}-\alpha$ is shown in Fig. 5. One notable difference from the other cases is that the Poincaré section now extends to infinity in $p_{r}$ as $r$ approaches zero. This is due to the attractive Coulomb force, which allows the particles to come arbitrarily close to each other.

No chaos was found for case of $\bar{d}-\alpha$. This is an interesting case because the gyrofrequencies sum to zero, so $q_{1} / m_{1}+q_{2} / m_{2}=0$ or equivalently $q=2-m$. To answer the question posed by Pinheiro and MacKay about what happens in this case, a search for chaos in general cases with $q=2-m$ was made. No cases that exhibited chaos were found, which suggests that the solutions are all periodic or quasiperiodic.

An automated search algorithm was used to find chaotic solutions with $m$ and $q$ in the interval $[1,2]$. In each case, initial conditions were taken randomly from a Gaussian


Figure 5: The projected Poincaré section for the two-body system of an antiproton and alpha particle taken at $\phi=0$. The color of each point indicates the $p_{\phi}$ position. Note that as $r \rightarrow 0, p_{r} \rightarrow \infty$.


Figure 6: A plot showing the regions where positive Lyapunov exponents were measured by an automated search algorithm that varied $m$ and $q$ with random initial conditions. These regions are indicated by darkened pixels. There is an observed lack of chaos along the line $q=m$, which is an integrable case. Also, no chaos was found along the line $q=2 m-1$.
distribution. A plot marking the locations where positive Lyapunov exponents were measured is shown in Fig. 6. There is a prominent lack of chaotic solutions on the line $q=m$, where solutions are integrable [10]. There is also a visible lack of chaotic solutions on the line $q=2 m-1$, suggesting that those cases may also be integrable. However, points nearby both of these lines are often chaotic, suggesting that these integrable solutions are a set of measure zero.

## 4 Conclusion

The classical system of the two-body Coulomb problem in a uniform magnetic field was investigated numerically in the restricted case of planar particles with an orthogonal magnetic field. The goal was to determine under which conditions the system can exhibit chaos. The analysis was done in a four-dimensional phase space with values of mass and charge chosen to represent common physical particles.

Chaos was confirmed by computing a positive largest Lyapunov exponent and observing a chaotic sea in the Poincaré section. For the case of charges with equal signs, which has been largely unexplored in the past, we found several new chaotic solutions. However, these cases may be of limited physical interest since the charges would repel in the third dimension if perturbed out of the plane. For the case of charges with opposite signs, the largest Lyapunov exponent would not converge accurately due to numerical issues, but a Poincaré section showed that chaos occurs in the antideuteron-alpha particle case. The observation of chaos in this case is consistent with the analytical study of Pinheiro and MacKay [10], and can possibly be studied in greater detail if collisions are regularized. An automated search for chaotic solutions was performed for the problem where gyrofrequencies sum to zero, and no chaotic solutions were detected. Finally, the location of chaotic solutions in the region $1<m<2$ and $1<q<2$ were studied. There were two prominent regions with no chaotic solutions, along the lines $q=m$ and $q=2 m-1$. The first of these is known to be integrable, but it is not clear why the second criterion would preclude chaos. Future analytical study of this case could be interesting, but is outside of the scope of the current paper.

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# On a Conjecture of Trichotomy and Bifurcation In a Third Order Rational Difference Equation 

Xianyi Li<br>College of Mathematical Science<br>Yangzhou University, Yangzhou<br>Jiangsu 225002, P. R. China.<br>e-mail: mathxyli@yzu.edu.cn<br>Cheng Wang<br>College of Mathematical Science<br>Yangzhou University, Yangzhou<br>Jiangsu 225002, P. R. China.<br>e-mail: mathwc@126.com


#### Abstract

In this paper it is first investigated for a conjecture of trichotomy of period two for a third order rational difference equation, and then the bifurcation of this equation is further considered. The results obtained partially verify a conjecture in a known literature.


Keywords: Rational difference equation, Trichotomy of period two, Global asymptotic stability, Center manifold; Bifurcation.

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## 1 Introduction and preliminaries

Consider the following third order rational difference equation

$$
\begin{equation*}
x_{n+1}=\frac{x_{n-1}+p}{x_{n-2}+q}, \quad n=0,1, \cdots, \tag{1}
\end{equation*}
$$

where the parameters $p, q$ are non-negative real numbers, and the initial conditions $x_{-2}, x_{-1}, x_{0}$ are positive real numbers.

For Eq. (1), M. R. S. Kulenovic and G. Ladas presented the following question:
Conjecture [3, P. 195] Assume that

$$
p, q \in[0, \infty) .
$$

(a) Show that every positive solution of the equation

$$
x_{n+1}=\frac{x_{n-1}+p}{x_{n-2}+q}, \quad n=0,1, \cdots,
$$

converges to a period-two solution if only and if $q=1$.
(b) Show that when $q>1$ the positive equilibrium of the equation is global asymptotical stable.
(c) Show that when $q<1$ the equation possesses positive unbounded solutions.

This question essentially is a conjecture for trichotomy of period two solution. Motivated by this question, our main aim in this paper is to investigate the global behavior of all positive solutions of Eq. (1).

The equilibrium point $\bar{x}$ of Eq. (1) satisfies

$$
\bar{x}=\frac{\bar{x}+p}{\bar{x}+q},
$$

i.e., $\bar{x}^{2}+(q-1) \bar{x}-p=0$. From this, one can see that Eq. (1) has a unique non-negative equilibrium point

$$
\begin{equation*}
\bar{x}=\frac{1-q+\sqrt{(1-q)^{2}+4 p}}{2} . \tag{2}
\end{equation*}
$$

The linearized equation of Eq. (1) associated with this equilibrium is

$$
\begin{equation*}
y_{n+1}-\frac{1}{\bar{x}+q} y_{n-1}+\frac{\bar{x}}{\bar{x}+q} y_{n-2}=0 \tag{3}
\end{equation*}
$$

with the characteristic equation

$$
\begin{equation*}
\lambda^{3}-\frac{1}{\bar{x}+q} \lambda+\frac{\bar{x}}{\bar{x}+q}=0 . \tag{4}
\end{equation*}
$$

For $q=1$, the unique non-negative equilibrium point of Eq. (1) reads $\bar{x}=\sqrt{p}$. Eq. (3) and Eq. (4) are respectively reduced into

$$
\begin{equation*}
y_{n+1}-\frac{1}{\sqrt{p}+1} y_{n-1}+\frac{\sqrt{p}}{\sqrt{p}+1} y_{n-2}=0 \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
(\lambda+1)\left[\left(\lambda-\frac{1}{2}\right)^{2}+\frac{3 \sqrt{p}-1}{4(\sqrt{p}+1)}\right]=0 . \tag{6}
\end{equation*}
$$

Eq. (6) always has one real root $\lambda_{1}=-1$, which denotes a period two solution $y_{n}=(-1)^{n}$ of the linearized equation (5) and corresponds to a one dimensional local center manifold $W_{\text {loc }}^{c}$ of Eq. (1).

For $p=0$, three roots of Eq. (6) are $-1,0,1$. The characteristic root $\lambda=0$ corresponds to a one dimensional (1D) local stable manifold $W_{l o c}^{s}$ of Eq. (1) whereas the unit roots $\lambda= \pm 1$ correspond to a two dimensional (2D) local center manifold $W_{l o c}^{c}$ of Eq. (1).

When $p \in\left(0, \frac{1}{9}\right]$, Eq. (6) has another two real roots

$$
\lambda_{2,3}=\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1-3 \sqrt{p}}{\sqrt{p}+1}}
$$

with

$$
\left|\lambda_{2,3}\right|=\left|\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1-\sqrt{3 p}}{\sqrt{p}+1}}\right|<1
$$

For $p \in\left(\frac{1}{9}, \infty\right)$, Eq. (6) has a pair of conjugate imaginary roots

$$
\lambda_{2,3}=\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{3 \sqrt{p}-1}{\sqrt{p}+1} i}
$$

satisfying

$$
\left|\lambda_{2,3}\right|=\left|\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\sqrt{p}}{\sqrt{p}+1}} i\right|=\frac{1}{2} \sqrt{1+\frac{\sqrt{p}}{\sqrt{p}+1}}<1
$$

So, for $p \in(0, \infty)$, Eq. (1) always has a 2D local stable manifold $W_{l o c}^{s}$.
There is always a point of view in engineers and physicians that the local stability of an equilibrium point in a given system implies its global stability.

If this conjecture is true, then the point of view is partly verified. But, we now see that this point of view is not always true.

Because every solution of Eq. (5) converges to either $\bar{x}$ or period two solution, it is natural to conjecture that every solution of Eq. (1) converges to a period two solution for $q=1$.

When the conjecture is true, the essential changes for the properties of solutions of Eq. (1) will take place at $q=1$. Namely, the bifurcation of Eq. (1) will occur at $q=1$. So, the parameter $q=1$ is a critical point (or bifurcation point).

Generally speaking, given a difference equation

$$
x_{n+1}=f\left(x_{n}, \mu\right), \quad n=0,1,2, \cdots,
$$

where $x_{n} \in R^{m}, \mu \in R^{k}, f \in C\left(R^{m+k}, R^{m}\right), m, k \in\{1,2, \cdots\}$ and the initial value $x_{0} \in R^{m}$, its solution is a continuous function with respect to the initial value $x_{0}$ and the parameter $\mu$, denoted by $x_{n}=x\left(n, x_{0}, \mu\right)$. If the change of the initial value $x_{0}$ or the parameter $\mu$ around a value leads to the essential change of the trajectory structure rule of its solution, then it is implied that a bifurcation of this equation occurs. Correspondingly, the critical value is called to be a bifurcation value. This is similar to the bifurcation of ordinary differential equation.

Certainly, it should be pointed out that the essential change of the trajectory structure rule of a difference equation contains many cases, such as, a solution or an invariant set
changes its number, its stability, its boundedness, its period or the cycle length, etc. Therefore, it is meaningful to investigate the bifurcation theory of difference equation according to its own right.

The study of rational difference equation (for short, RDE ) is quite challenging and rewarding due to the fact that some results of RDEs offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations; moreover, the investigations of RDEs are still in its infancy so far. To see this, refer to the monographs [1-4] and the papers [5-14] and the references cited therein. Especially, we solved an open problem for the boundedness for the a generalized RDE in [13]; we obtained a result for the global asymptotical stability of a kind of RDE. As a special case, our results solve a conjecture for the global asymptotical stability of a RDE [14].

This rest of this paper is arranged as follows. The global asymptotical stability for the positive equilibrium of Eq. (1) with $q>1$ is shown in Section 2, which thoroughly solves Conjecture (b).

The investigation for period-two solution of Eq. (1) is formulated in Section 3, partially answering Conjecturer (a) and leaving a gap of $p \in(0,1)$ to readers. It is shown in Section 4 that this equation possesses positive unbounded solutions when $q<1$, completely solving Conjecture (c). Synthesizing these results, the center manifold for the equilibrium of Eq. (1) and the analysis of bifurcation are stated in the final Section 5.

## 2 Global asymptotic stability of positive equilibrium point

For the global asymptotical stability of positive equilibrium point of Eq. (1), one has the following results.

Theorem 2.1 The positive equilibrium point of Eq. (1) is global asymptotical stable for $q>1$.

Proof For $q>1$, the linearized equation of Eq. (1) associated with the positive equilibrium (2) is

$$
y_{n+1}-\frac{1}{\bar{x}+q} y_{n-1}+\frac{\bar{x}}{\bar{x}+q} y_{n-2}=0 .
$$

Obviously, $\left|-\frac{1}{\bar{x}+q}\right|+\left|\frac{\bar{x}}{\bar{x}+q}\right|=\frac{1+\bar{x}}{\bar{x}+q}<1$. So, by [2, Remark 1.3.1, P. 13], the positive equilibrium of Eq. (1) is locally asymptotically stable for $q>1$.

Next, one will show the positive equilibrium of Eq. (1) is globally attractive for $q>1$. In view of Eq. (1), one can see

$$
\begin{equation*}
x_{n+1}<\frac{1}{q} x_{n-1}+\frac{p}{q}, n=0,1, \cdots . \tag{7}
\end{equation*}
$$

Denote $n=2 s+t, t \in\{0,1\}, y_{s}=x_{2 s+t}$ and $r=\frac{1}{q} \in(0,1)$.
Then it follows from (7) that

$$
\begin{equation*}
y_{s+1}<r y_{s}+r p, s=0,1, \cdots . \tag{8}
\end{equation*}
$$

So, one further gets from (8)

$$
y_{s+1}<r^{s+1} y_{0}+r p+p r^{2}+p r^{3}+\cdots+p r^{s+1}=r^{s+1} y_{0}+p \frac{r-r^{m}}{1-r}<y_{0}+\frac{p r}{1-r}
$$

which indicates that $y_{s}$ possesses upper bound, say, $R=y_{0}+\frac{p r}{1-r}$. And hence so does $\left\{x_{n}\right\}$. Accordingly, from Eq. (1), $x_{n+1}>\frac{p}{R+q}$, namely, $\left\{x_{n}\right\}$ has lower bound.

Therefore,

$$
\lim _{n \rightarrow \infty} \text { infx }_{n}=L \quad \text { and } \quad \lim _{n \rightarrow \infty} \sup x_{n}=M
$$

exist and are finite.
Moreover,

$$
0<L \leq \bar{x} \leq M<\infty
$$

It is clear from Eq. (1) that

$$
L \geq \frac{L+p}{M+q}, \quad M \leq \frac{M+p}{L+q} .
$$

So, $L M+L q \geq L+p$ and $M L+M q \leq M+p$, which implies $(1-q) M+p \geq M L \geq$ $(1-q) L+p$. Therefore, $M \leq L$. Again, $M \geq L$. Hence $M=L$. That is to say, $\lim _{n \rightarrow \infty} x_{n}=\bar{x}$. The proof is complete.

## 3 Existence of period two solution

In this section one will consider the behavior for prime period two solutions of Eq. (1). The following lemma will be needed [4, P. 12].

Lemma 3.1 Let $F \in C\left[I^{k}, I\right]$ for some interval $I$ of positive real numbers and for some natural number $k$. Then every positive solution of the equation

$$
x_{n}=F\left(x_{n-1}, \cdots, x_{n-k}\right), n=0,1, \cdots
$$

has a limit in $I$ if the following statements are true:
(1) $F\left(z_{1}, \cdots, z_{k}\right)$ is nondecreasing in each of its arguments;
(2) $F\left(z_{1}, \cdots, z_{k}\right)$ is strictly increasing in each of the arguments $z_{i_{1}}, \cdots, z_{i_{e}}$, where $i_{1}, \cdots, i_{e}$ are relatively prime;
(3) $F(c, c, \cdots, c)=c$ for every $c \in I$.

One has the following result.
Theorem 3.2 (1) If every positive solution of Eq. (1) converges to a period two solution, then $q=1$.
(2) If $q=1$, moreover, $p \in\{0\} \cup[1, \infty)$, then every positive solution of Eq. (1) converges to a period two solution.

Proof (1) If every positive solution $\left\{x_{n}\right\}_{n=-2}^{\infty}$ of Eq. (1) converges to a period two solution $\cdots, \alpha, \beta, \alpha, \beta, \alpha, \cdots$, then

$$
\alpha=\frac{\alpha+p}{\beta+q}, \quad \beta=\frac{\beta+p}{\alpha+q} .
$$

Namely,

$$
\left\{\begin{array}{l}
\alpha \beta+q \alpha=\alpha+p  \tag{9}\\
\alpha \beta+q \beta=\beta+p
\end{array}\right.
$$

Subtracting each other in the above system yields $q(\alpha-\beta)=\alpha-\beta$. Notice that $\alpha \neq \beta$. So, $q=1$.
(2) Assume $q=1$. First consider the case $p=0$. Then Eq. (1) reads

$$
\begin{equation*}
x_{n+1}=\frac{x_{n-1}}{x_{n-2}+1}<x_{n-1}, n=0,1,2, \cdots \tag{10}
\end{equation*}
$$

Set $n=2 s+t, y_{s}=x_{2 s+t}, t \in\{0,1\}$. Then (10) implies $0<y_{s+1}<y_{s}$. Therefore, $\lim _{s \rightarrow \infty} y_{s}$ exists, which implies that both $\left\{x_{2 s}\right\}$ and $\left\{x_{2 s+1}\right\}$ converge. Denote

$$
\lim _{n \rightarrow \infty} x_{2 n}=\alpha, \lim _{n \rightarrow \infty} x_{2 n+1}=\beta .
$$

Letting $n$ in Eq. (1) be changed into $2 n$ and $2 n+1$ respectively and then respectively taking the limits on both sides of Eq. (1) yield

$$
\alpha=\frac{\alpha+p}{\beta+q}, \quad \beta=\frac{\beta+p}{\alpha+q} .
$$

So, $\cdots, \alpha, \beta, \alpha, \beta, \cdots$ is a period two solution of Eq. (1). Accordingly, $\left\{x_{n}\right\}$ converges to a period two solution (not necessarily prime).

Then consider the case $p \in[1, \infty)$. From Eq. (1), one has $x_{n}=\frac{x_{n-2}+p}{x_{n-3}+1}$ and so $x_{n-5}=\frac{x_{n-4}+p}{x_{n-2}}-1$.Thus,

$$
\begin{equation*}
x_{n+2}=\frac{x_{n}+p}{x_{n-1}+1}=\frac{x_{n}+p}{\frac{x_{n-3}+p}{x_{n-4}+1}+1}=\frac{x_{n}+p}{\frac{x_{n-5}+p}{x_{n-6}+1}+p} x_{n-4}+1 \quad \frac{x_{n}+p}{\frac{\frac{x_{n-}+p}{x_{n-2}-1+p}}{\frac{x_{n-6}+1}{x_{n-4}+p}}+1} . \tag{11}
\end{equation*}
$$

Put $n=2 s+t, t \in\{0,1\}$ and $y_{s}=x_{2 s+t}, s=-1,0, \cdots$.. Then, from (11), one can see

$$
\begin{equation*}
y_{s+1}=\frac{y_{s}+p}{\frac{\frac{y_{s-2}+p}{\frac{y_{s-1}}{}-1+p}}{y_{s-3}+1}+p} y_{s-2+1}^{y_{s}+1}+1 \text {. } \tag{12}
\end{equation*}
$$

Evidently, $H$ is increasing with respect to $y_{s}, y_{s-1}$, and $y_{s-3}$. Furthermore, $H(x, x, x, x)=$ $x$ for any $x \in(0, \infty)$.

Next one will show that $H$ is increasing in $y_{s-2}$. Denote

$$
h(x)=\frac{\frac{\frac{x+p}{y}-1+p}{z+1}+p}{x+1}, x, y, z \in(0, \infty) .
$$

Then $h^{\prime}(x)=\frac{(1-p)(1+y)-p y(z+1)}{y(z+1)(x+1)^{2}}<0$ for $p \geq 1$. That is to say, $h(x)$ is decreasing in $x$. Thereout, $H$ is increasing with respect to $y_{n-2}$ for $p \geq 1$. Hence, it follows from Lemma 3.1 that every solution $\left\{y_{s}\right\}_{s=-1}^{\infty}$ has a limit and so $\left\{x_{2 s}\right\}_{s=0}^{\infty}$ and $\left\{x_{2 s+1}\right\}_{s=-1}^{\infty}$ converge, which indicates that $\left\{x_{n}\right\}$ converges to a period two solution for $q=1$ and $p \geq 1$. The proof is complete.

Remark 3.3. If it can be proved that every positive solution of Eq. (1) converges to a period two solution for $p=1$ and $q \in(0,1)$, then Conjecture (a) will be completely shown. Unfortunately, up to now, this is still an open problem.

## 4 Existence of unbounded solution

In this section one will investigate the existence of unbounded solutions of Eq. (1) for $q<1$. The following results are derived.

Theorem 4.1 There exist unbounded solutions of Eq. (1) for $q<1$.
Proof Consider two cases.
Case 1: $p>0$. Choose the initial values $x_{0}, x_{-2} \in(0,1-q), x_{-1} \geq \frac{p}{x_{0}}+1-q$, which implies $\frac{x_{0}+p}{x_{-1}+q} \leq x_{0}$. From $x_{n+1}=\frac{x_{n-1}+p}{x_{n-2}+q}, n=0,1,2, \cdots$, one has

$$
\begin{array}{ll}
x_{1}=\frac{x_{-1}+p}{x_{-2}+q}>x_{-1}+p, & x_{2}=\frac{x_{0}+p}{x_{-1}+q} \leq x_{0}, \\
x_{3}=\frac{x_{1}+p}{x_{0}+q}>x_{1}+p>x_{-1}+2 p, & x_{4}=\frac{x_{2}+p}{x_{3}+q}<\frac{x_{0}+p}{x_{1}+q}<x_{0}, \\
x_{5}=\frac{x_{3}+p}{x_{2}+q} \geq \frac{x_{3}+p}{x_{0}+q}>x_{3}+p>x_{-1}+3 p, & x_{6}=\frac{x_{4}+p}{x_{3}+q}<\frac{x_{0}+p}{x_{1}+q}<x_{0} .
\end{array}
$$

So, inductively, one gets

$$
x_{2 n+1}>x_{-1}+(n+1) p, \quad x_{2 n}<x_{0}, n=0,1, \cdots .
$$

Therefore, $\lim _{n \rightarrow \infty} x_{2 n+1}=\infty$, i.e., $\left\{x_{n}\right\}$ is unbounded.
Case 2: $p=0$. Then by choosing the initial values $x_{0}, x_{-1}, x_{-2} \in(0, \infty)$ such that $0<x_{0}<x_{-2}<1-q, x_{-1}>1-q$, one has

$$
\begin{array}{rlrl}
x_{1} & =\frac{x_{-1}}{x_{-2}+q}>x_{-1}, & x_{2} & =\frac{x_{0}}{x_{-1}+q}<x_{0} \\
x_{3} & =\frac{x_{1}}{x_{0}+q}>\frac{x_{1}}{x_{-2}+q}=\left(\frac{1}{x_{-2}+q}\right)^{2} x_{-1}, & x_{4} & =\frac{x_{2}}{x_{1}+q}<\frac{x_{0}}{x_{-1}+q}<x_{0} \\
x_{5} & =\frac{x_{3}}{x_{2}+q}>\frac{x_{3}}{x_{-2}+q}>\left(\frac{1}{x_{-2}+q}\right)^{3} x_{-1}, & x_{6}=\frac{x_{4}}{x_{3}+q}<\frac{x_{0}}{x_{-1}+q}<x_{0} .
\end{array}
$$

It follows by induction that $x_{2 n+1}>\left(\frac{1}{x_{-2}+q}\right)^{n} x_{-1}$ and so $\lim _{n \rightarrow \infty} x_{2 n+1}=\infty$, which implies that $\left\{x_{n}\right\}$ is also unbounded. The proof is over.

## 5 Analysis of bifurcation

After the above preparations, one will begin to formulate some results for the center manifold of the equilibrium of Eq. (1) and analyze the bifurcation case of Eq. (1).

First, one may transform Eq. (1) to an equivalent system. Let $u_{n}=x_{n-2}, v_{n}=$ $x_{n-1}, w_{n}=x_{n}$, and $z_{n}=\left(u_{n}, v_{n}, w_{n}\right)^{T}$. Then Eq. (1) is equivalent to the following system: $z_{n+1}=F\left(z_{n}\right)$, i.e.,

$$
\left\{\begin{align*}
u_{n+1} & =v_{n},  \tag{13}\\
v_{n+1} & =w_{n}, \\
w_{n+1} & =\frac{v_{n}+p}{u_{n}+q} .
\end{align*}\right.
$$

The equilibrium point $\bar{z}=(u, v, w)$ of the system (13) satisfies

$$
u=v=w=\bar{x}=\frac{1-q+\sqrt{(1-q)^{2}+4 p}}{2} .
$$

The Jaccobian matrix of $F$ at the equilibrium point $\bar{z}$ has the form

$$
D F(\bar{z})=\left(\begin{array}{lll}
o & 1 & 0 \\
o & 0 & 1 \\
-\frac{v+p}{(u+q)^{2}} & \frac{1}{u+q} & 0
\end{array}\right)
$$

with the characteristic equation evaluated at the equilibrium point $\bar{z}$

$$
\begin{equation*}
\lambda^{3}-\frac{1}{\bar{x}+q} \lambda+\frac{\bar{x}}{\bar{x}+q}=0, \tag{14}
\end{equation*}
$$

which is the same as (4).
The following results may be derived.
Theorem 5.1 Consider the first order 3D system (13). Then the following statements are true.

1. Suppose $q=1$. If $p=0$, then the equilibrium point $\bar{z}$ of the system (13) is a center with a 1D local stable manifold and a 2D local center manifold; If $p>0$, there always is a 2D local stable manifold and a 1D local center manifold at the neighborhood of equilibrium point $\bar{z}$; the latter is a segment of curve $L$ consisting of $\bar{z}$ and the total period two solutions of $F$, where $L=\left\{(u, v, w) \in\left(R^{+}\right)^{3} \mid u v=p, u=w\right\}$, and it is globally asymptotically stable, i.e., the other solutions of (13) regard $L$ as a limit set.
2. For $q>1$, the equilibrium point $\bar{z}$ of the system (13) is a stable one. There is a 3D stable manifold at the neighborhood of the equilibrium point $\bar{z}$. Namely, the center manifold (2D for $p=0$ and 1 D for $p>0$ ) which occurs for $q=1$ disappears and turns also into a stable manifold (2D for $p=0$ and 1D for $p>0$ ).
3. For $q<1$, the center manifold becomes an unstable manifold. At this time, except for the orbit $z_{n}=\bar{z}$ in $L$, all other orbits on $L$ will tend to infinity along the $L$.

Proof It is easy to see that the first order 3D system (13) has a unique equilibrium point $\bar{z}$.

1. For $q=1$, according to the analysis in the introduction in this paper, the characteristic equation (14) of the system (13) has one root $\lambda_{1}=-1$. When $p=0$, the other two roots of (14) is 0,1 . The characteristic root 0 corresponds a 1D local stable manifold of the equilibrium point $\bar{z}$ of the system (13), and the characteristic roots $\pm 1$ correspond a 2 D local center manifold of the equilibrium point $\bar{z}$. When $p \in\left(0, \frac{1}{9}\right]$, the equation (14) has two other real roots

$$
\lambda_{2,3}=\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1-3 \sqrt{p}}{\sqrt{p}+1}}
$$

with $\left|\lambda_{2,3}\right|<1$. When $p \in\left(\frac{1}{9}, \infty\right)$, the equation (14) has a pair of conjugate imaginary roots

$$
\lambda_{2,3}=\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{3 \sqrt{p}-1}{\sqrt{p}+1} i}
$$

satisfying

$$
\left|\lambda_{2,3}\right|=\left|\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\sqrt{p}}{\sqrt{p}+1}} i\right|=\frac{1}{2} \sqrt{1+\frac{\sqrt{p}}{\sqrt{p}+1}}<1
$$

Hence, it is always true that $\left|\lambda_{2,3}\right|<1$ for $p \in(0, \infty)$, which, together with $\lambda_{1}=-1$, reads the existence of a 2 D local stable manifold and a 1 D local center manifold at the neighborhood of equilibrium point $\bar{z}$. The expression of $L$ can be obtained from $z_{n+2}=z_{n}$.
2. For $q>1$, the previous Theorem 2.1 tells us the equilibrium point $\bar{z}$ of the system (13) is globally asymptotically stable regardless of $p=0$ or $p>0$. This indicates that there is a 3D stable manifold at the neighborhood of the equilibrium point $\bar{z}$. Therefore, the 2D center manifold which occurs for $p=0$ and $q=1$ disappears and turns into a 2D stable manifold and the 1D center manifold occurring for $p>0$ and $q=1$ disappears and becomes a 1D stable manifold.
3. The correctness follows from Theorem 4.1 stated previously in this paper.

Remark 5.2. It is easily observed that the equilibrium point $\bar{z}$ of the system (13) loses one dimension in the center manifold and gains one dimension in the stable one when $p=1$ and $q$ crosses the null value. Namely, the dimensional number of center manifold of the equilibrium point $\bar{z}$ of the system (13) varies from 2 to 1 . This kind of change for the dimensional number of center manifold of the equilibrium point $\bar{z}$ as the parameter $p$ crosses the null value possibly implies a new mechanism for the creation of bifurcation, which deserves to one's further investigations.

## 6 Stability of period two solution

The existence of period two solution has been considered for $q=1$ in above Section 3 . When every solution of Eq. (1) converges to a period two solution, how about the stability of the period two solution? We now answer this question.

A period two solution of Eq. (1) or system (13) is a fixed point of $F^{2}(z)=F(F(z))=z$ with

$$
F^{2}(z)=F(F(z))=\left(\begin{array}{c}
w \\
\frac{v+p}{u+q} \\
\frac{w+p}{v+q}
\end{array}\right) \quad \text { for } \quad z=\left(\begin{array}{c}
u \\
v \\
w
\end{array}\right)
$$

The Jaccobian matrix of $F^{2}$ has the form

$$
D F^{2}(z)=\left(\begin{array}{lcc}
0 & 0 & 1 \\
-\frac{v+p}{(u+q)^{2}} & \frac{1}{u+q} & 0 \\
0 & -\frac{w+p}{(v+q)^{2}} & \frac{1}{v+q}
\end{array}\right)
$$

with the characteristic equation evaluated at the period two solution $z$

$$
\begin{equation*}
\lambda^{3}-\left(\frac{1}{u+q}+\frac{1}{v+q}\right) \lambda^{2}+\frac{\lambda}{(u+q)(v+q)}-\frac{w+p}{(u+q)^{2}(v+q)}=0 . \tag{15}
\end{equation*}
$$

By Theorem 3.2 and Theorem 5.1, one can see that there exist period two solutions of Eq. (1) or system (13) only when $q=1$ and that the period two solution $z^{T}=(u, v, w) \in\left(R^{+}\right)^{3}$
satisfies $u v=p$ and $w=u$. Hence, Eq. (15) can be reduced to

$$
\begin{equation*}
\lambda^{3}-\left(\frac{1}{u+1}+\frac{u}{u+p}\right) \lambda^{2}+\frac{u}{(u+1)(u+p)} \lambda-\frac{u p}{(u+1)(u+p)}=0 \tag{16}
\end{equation*}
$$

where $u>0$ is a parameter.
Theorem 6.1 Any one period two solution of Eq. (1) with $q=1$ is unstable.
To prove this conclusion, the following lemma is needed [3, P. 46].
Lemma 6.2 For the equation $\lambda^{3}+a \lambda^{2}+b \lambda+c=0$ with real coefficients $a, b, c$, all roots lie inside the unit disk $|\lambda|<1$ if and only if $|a+c|<1+b,|a-3 c|<3-b$ and $b+c^{2}<1+a c$.

Proof of Theorem 6.1 Corresponding to (16), the condition

$$
\begin{aligned}
|a+c|<1+b & \Leftrightarrow \frac{1}{u+1}+\frac{u}{u+p}+\frac{u p}{(u+1)(u+p)}<1+\frac{u}{(u+1)(u+p)} \\
& \Leftrightarrow u+p+u(u+1)+u p<(u+1)(u+p)+u \\
& \Leftrightarrow 0<0 .
\end{aligned}
$$

This is impossible. Hence, there exist at least one root of Eq. (16) not to lie inside the unit disk $|t|<1$. Thus, Any one period two solution of Eq. (1) with $q=1$ is unstable.

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# Book Review <br> Lozi Mappings: Theory and Applications by Zeroualia Elhadj, CRC Press, 2013 

René Lozi<br>Laboratory of Mathematics J. A. Dieudonné, U.M.R. C.N.R.S 7351, University of Nice-Sophia Antipolis, France.<br>e-mail: R.LOZI@unice.fr


#### Abstract

When, more than two years ago, Prof. Zeroualia Elhadj informed me of his willing to write a book on what is known as "Lozi map" since the Misiurewicz's communication in the congress organized by the New York Academy of Science, 17-21 December 1979, I warned the task was not straightforward because hundreds of articles were published on this topic in thirty years. These papers were scattered in various fields of research, not only in mathematics (dynamical systems), but in physics, computer science, electronics, chemistry, control science and engineering, etc.


Keywords: Lozi Mappings, mathematical theory, real world-applications

Nevertheless, he eventually collected and scrutinized more than one thousand papers before completing this outstanding book "Lozi Mappings: Theory and Applications". The outcome of his enquiry is tremendous. Every aspect of the mathematical properties of this map of the plane (and its generalizations) is analyzed. The results are classified and systematized. Moreover, in order to make easy the comprehension for a fresh reader, the book begins with a comprehensive review of hyperbolicity, ergodicity and chaos. Once the background is clearly posed the reality of chaos in the Hénon mappings is examined, after that the survey on Lozi mappings begins.

Responding to the kind invitation of Prof. Zeroualia Elhadj to write an introduction, I take the opportunity to introduce some personal views not only on the matter of chaotic


Figure 1: The initial area a maped by $T^{\prime}$ into $b$, then by $T^{\prime \prime}$ into $c$, and finally by $T^{\prime \prime \prime}$ into $d$.
systems, but also on the current evolution of mathematics and some aspects of the live of one researcher in mathematics.

In the human life it is not so easy to recall a particular day. Thirty five years after the pinpoint moment I had the idea to substitute the quadratic term in the Hénon map by an absolute value I can remember the exact date because it took place during the talk of the presentation of the thesis of A. Intissar on June 15th 1977 around 11 a.m (I checked recently the date). In these days the department of mathematics of the university of Nice (later called university of Nice-Sophia Antipolis) was a small community and every one attended the presentation of each Ph.D. thesis. Hence I was not very concerned by the talk and contrarily I was thinking thoroughly to the strange structure of the Hénon map that my colleague Gérard Iooss told me about, few days before, during the "International Conference on Mathematical Problems in Theoretical Physics" that took place at the university of Roma, Italy (June 6-15), we partially attended together. The opening talk of this conference given by David Ruelle (Dynamical Systems and Turbulent Behavior) emphasized the importance of such a simple discrete model in the study of turbulence (this is not recognized today). At that time I occupied the position of "Attaché de Recherches" at C.N.R.S. (Centre National de la Recherche Scientifique) after my Ph. D. thesis on numerical analysis of bifurcation problems (the first thesis on bifurcation theory in France, presented on April 25th, 1975). I was mainly interested in discretization problems and finite element methods, in which nonlinear function are approximated by piecewise linear ones. I tried to apply my background to the quadratic map introduced by Michel Hénon few months ago, in order to obtain a better amenable map for analytical treatment. In Figure 1 of his publication (reproduced here) there is a clear explanation of the folding and stretching process which led him to the formula of the map.

The area $b$ on the Figure is bounded by two parabolas generated by the formula: $T^{\prime}: x^{\prime}=x, y^{\prime}=y+1-a x^{2}$ applied to the initial area $a$. Drawing on a paper sheet the shape of this area, I embedded it in another area bounded by four line segments which eventually reminded me the graph of the absolute value function. I substituted then $L^{\prime}: x^{\prime}=x, y^{\prime}=y+1-a|x|$ to $T^{\prime}$. Soon after the end of the presentation I went to my office situated on the upper floor of the seminar room to test the idea on the Hewlett-Packard 9820 calculator linked to the HP 9862 plotter I used to promote computer science for teachers in the classroom at the Institute of Research in Educational Mathematics (I.R.E.M). Even if the parameter value giving the "classical" Hénon strange
attractor (i.e., $a=1.4, b=0.3$ ) provides also a strange attractor for this new mappings, after few tests I shifted it to $a=1.7, b=0.5$ in order to obtain a more striking picture of the strange attractor studied in this book. Back to the lunch which celebrated the completion of the thesis I showed the figure to Gérard Iooss and also to Alain Chenciner who encouraged me latter to publish the formula, (the genuine article comes from the presentation I gave during a conference on dynamical systems in July 1977 in Nice). In the following days I was convinced that few weeks would be enough to explain the structure of such an attractor basically composed of line segments. But the task proved more difficult than expected (mainly because contrary to as Michal Misiurewicz did, I did not limited the extend of the parameter value for the study). In the next years I attended two meetings on iteration theory: the first one on May 21-23, 1979 at La Garde Freinet (a small town in the south of France) where Michel Hénon was also present and where Michal Misiurewicz, after some questions at the end of my talk (the purpose of which was the computation of homoclinic points of the map), came to the blackboard to give to the assistance some clues of the forthcoming result of the New York meeting. The second meeting is a summer school in physics on July 1979 in Cargèse (Corsica) in the proceedings of which I eventually published the article entitled: "Strange attractors: a class of mappings of $\mathbb{R}^{2}$ which leaves some Cantor Set invariant". In this paper I used the genuine non differentiable map in order to prove the existence of one homoclinic point for a smooth version of the Lozi map and then applying a theorem of Stephen Smale I proved the existence of an invariant Cantor set. After that, took place the congress organized by the New-York Academy of Science where I am proud I shook the hand of Edward Lorenz, the father of strange attractors and I listen with a mix of anxiety and curiosity the first proof of existence of a strange attractor for an analytically given map of the plane. After the Misiurewicz's work, hundreds of papers were published on countless aspects of this strange attractor as it is cleverly showed in this book.

Now, if we go back in thought in the late 70 's, in some aspects, life was very different than nowadays. There was no personal computer (M. Hénon used one of the only two computers of the university of Nice, a IBM 7040, in order to plot the figure of his original paper), no Internet, no wireless phone. Also, communications between researchers were done through slow post office mail, travels by air were very expensive, limiting personal contact between researchers in the west countries. Moreover, the Berlin's wall was still standing. James Yorke which coined the term of chaos in his famous paper with his student Tien-Yien Li "Period three implies Chaos" in 1975 was unaware of Alexander Sharkovkii's theorem published in Russian in 1964 displaying more penetrating results on periodic orbits (however essential notion as sensitive dependence on initial conditions is only introduced in the paper of Li and Yorke). The technical progress in thirty years is dramatic in every aspect of all the days life. In contrast, mathematics is progressing very slowly. Near my entire professional life of mathematician has been needed to see published results I expected proved in few months. News results as for example:"topological entropy for the Lozi maps can jump from zero to a value above 0.1203 as one crosses a particular parameter and hence it is not upper semi-continuous in general" (I. B. Yildiz), or: "certain Lozi-like maps have the orbit-shifted shadowing property" (A. Sakurai) are continuously published every year.

A tentative title first chosen to the book by Prof. Elhadj Zeraoulia: "the power of
chaos" reflects a part of what one can observe in numerous publications of last years. If on one hand new theoretical important results are still regularly found as said above, on the other hand applications of Lozi map are soaring, in engineering, computers, communications, control, medicine and biology and especially in the domain of evolutionary algorithms. In the last chapter of the book, due to the limitation of the number of pages, only some real-world applications are given. However, they allow the reader to get an idea of the power that holds the mastery of chaos generated by the Lozi mappings.

In the scope of evolutionary algorithms the use of chaotic sequences instead of random ones has been introduced ten years ago by Caponetto, et al. Several traditional chaotic maps in 1 or 2 -dimensions are used. The difference among them is based on the two main differences: the shape of the invariant measure and the robustness with respect to numerical computation. Lozi map is recognized to show better performance due to his shadowing property. These algorithms are particularly efficient in global optimization problems. Since 1975, when I was a young assistant professor, I had a disappointed passion for years in the search of classical algorithm for solving such problems. What is funny nowadays is to notice how a map I introduced for an entire different aim is routinely used now to solve this problem.

Finally thanks to the work of Prof. Elhadj Zeroualia, I can go back and look at how a very small idea has blossomed into an area that still fascinates me: mathematics.

